

Extra Examples of ε and δ precise proofs for Limits

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Math 11

1. Prove : $\lim_{x \rightarrow 2} 7x - 6 = 8$ using the $\varepsilon - \delta$ definition of the limit.

Scratchwork: we want $|f(x) - L| = |(7x - 6) - 8| < \varepsilon$

$$\begin{aligned} |f(x) - L| &= |(7x - 6) - 8| = |7x - 14| = |7(x - 2)| = |7||x - 2| = 7|x - 2| \text{ (want } < \varepsilon) \\ 7|x - 2| < \varepsilon &\text{ means } |x - 2| < \frac{\varepsilon}{7} \end{aligned}$$

So choose $\delta = \frac{\varepsilon}{7}$ to restrict $0 < |x - 2| < \delta$. That is $0 < |x - 2| < \frac{\varepsilon}{7}$.

Proof: Let $\varepsilon > 0$ be given. Choose $\delta = \frac{\varepsilon}{7}$. Given x such that $0 < |x - 2| < \delta$, then

$$|f(x) - L| = |(7x - 6) - 8| = |7x - 14| = |7(x - 2)| = |7||x - 2| = 7|x - 2| < 7 \cdot \frac{\varepsilon}{7} = \varepsilon.$$

□

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2. Prove : $\lim_{x \rightarrow 1} 4 - 5x = -1$ using the $\varepsilon - \delta$ definition of the limit.

Scratchwork: we want $|f(x) - L| = |(4 - 5x) - (-1)| < \varepsilon$

$$\begin{aligned} |f(x) - L| &= |(4 - 5x) - (-1)| = |4 - 5x + 1| = |5 - 5x| = | - 5x + 5| = | - 5(x - 1)| = \\ &= | - 5||x - 1| = 5|x - 1| \text{ (want } < \varepsilon) \\ 5|x - 1| < \varepsilon &\text{ means } |x - 1| < \frac{\varepsilon}{5} \end{aligned}$$

So choose $\delta = \frac{\varepsilon}{5}$ to restrict $0 < |x - 1| < \delta$. That is $0 < |x - 1| < \frac{\varepsilon}{5}$.

Proof: Let $\varepsilon > 0$ be given. Choose $\delta = \frac{\varepsilon}{5}$. Given x such that $0 < |x - 1| < \delta$, then

$$\begin{aligned} |f(x) - L| &= |(4 - 5x) - (-1)| = |4 - 5x + 1| = |5 - 5x| = | - 5x + 5| = | - 5(x - 1)| \\ &= | - 5||x - 1| = 5|x - 1| < 5 \cdot \frac{\varepsilon}{5} = \varepsilon. \end{aligned}$$

□

3. Prove : $\lim_{x \rightarrow 12} \frac{x}{6} - 4 = -2$ using the $\varepsilon - \delta$ definition of the limit.

Scratchwork: we want $|f(x) - L| = \left| \left(\frac{x}{6} - 4 \right) - (-2) \right| < \varepsilon$

$$|f(x) - L| = \left| \left(\frac{x}{6} - 4 \right) - (-2) \right| = \left| \frac{x}{6} - 4 + 2 \right| = \left| \frac{x}{6} - 2 \right| = \left| \frac{1}{6}(x - 12) \right| = \left| \frac{1}{6} \right| |x - 12| = \frac{1}{6} |x - 12|$$

(want $< \varepsilon$)

$$\frac{1}{6} |x - 12| < \varepsilon \text{ means } |x - 12| < 6 \cdot \varepsilon$$

So choose $\delta = 6 \cdot \varepsilon$ to restrict $0 < |x - 12| < \delta$. That is $0 < |x - 12| < 6 \cdot \varepsilon$.

Proof: Let $\varepsilon > 0$ be given. Choose $\delta = 6 \cdot \varepsilon$. Given x such that $0 < |x - 12| < \delta$, then

$$\begin{aligned} |f(x) - L| &= \left| \left(\frac{x}{6} - 4 \right) - (-2) \right| = \left| \frac{x}{6} - 4 + 2 \right| = \left| \frac{x}{6} - 2 \right| = \left| \frac{1}{6}(x - 12) \right| \\ &= \left| \frac{1}{6} \right| |x - 12| = \frac{1}{6} |x - 12| < \frac{1}{6} \cdot 6 \cdot \varepsilon = \varepsilon. \end{aligned}$$

□

Please try and pay attention to the format of your proof. Be explicit about where your proof starts and ends. Be careful to follow the $\varepsilon - \delta$ definition of the limit. Show all of your work in order to justify your statements. Finally be clear on what your choice of δ is.