**Math 11 Final Examination May 11, 2011**

# Answer Key

**1.** [20 Points] Evaluate each of the following limits. Please justify your answers. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist.

(a) $\lim_{x\to 1} \dfrac{x^2-1}{(x+1)^2-1} = \lim_{x\to 1} \dfrac{0}{4-1} = \boxed{0}$ by DSP

(b) $\lim_{x\to 3^-} \dfrac{x^2-8x+15}{1-8x+g(x+1)}$, where $g(x) = x^2+7$.

$$\lim_{x\to 3^-} \frac{x^2-8x+15}{1-8x+g(x+1)} = \lim_{x\to 3^-} \frac{x^2-8x+15}{1-8x+(x+1)^2+7}$$

$$= \lim_{x\to 3^-} \frac{x^2-8x+15}{1-8x+(x^2+2x+1+7)} = \lim_{x\to 3^-} \frac{x^2-8x+15}{1-8x+(x^2+2x+8)}$$

$$= \lim_{x\to 3^-} \frac{x^2-8x+15}{x^2-6x+9} = \lim_{x\to 3^-} \frac{(x-5)(x-3)}{(x-3)(x-3)} = \lim_{x\to 3^-} \frac{x-5}{x-3} = \frac{-2}{0^-} = \boxed{+\infty}$$

(c) $\lim_{x\to 8} \dfrac{8-x}{\sqrt{x+1}-3} = \lim_{x\to 8} \dfrac{8-x}{\sqrt{x+1}-3} \cdot \dfrac{\sqrt{x+1}+3}{\sqrt{x+1}+3} = \lim_{x\to 8} \dfrac{(8-x)(\sqrt{x+1}+3)}{(x+1)-9}$

$$= \lim_{x\to 8} \frac{-(x-8)(\sqrt{x+1}+3)}{x-8} = \lim_{x\to 8} -(\sqrt{x+1}+3) = -(\sqrt{9}+3) = \boxed{-6}$$

(d) $\lim_{x\to 7} \dfrac{x^2-5x-14}{|7-x|}$ $\boxed{\text{DNE}}$ since RHL$\neq$LHL. see below.

Note: $|7-x| = \begin{cases} 7-x & \text{if } x \le 7, \text{ that is } 7-x \ge 0 \\ -(7-x) & \text{if } x > 7, \text{ that is } 7-x < 0 \end{cases}$

RHL: $\lim_{x\to 7+} \dfrac{x^2-5x-14}{|7-x|} = \lim_{x\to 7+} \dfrac{x^2-5x-14}{-(7-x)} = \lim_{x\to 7+} \dfrac{(x-7)(x+2)}{x-7} = \lim_{x\to 7+} x+2 = 9$

LHL: $\lim_{x\to 7-} \dfrac{x^2-5x-14}{|7-x|} = \lim_{x\to 7-} \dfrac{x^2-5x-14}{7-x} = \lim_{x\to 7-} \dfrac{(x-7)(x+2)}{7-x} = \lim_{x\to 7-} -(x+2) = -9$

**2.** [30 Points] Compute each of the following derivatives. Simplify numerical answers. Do not simplify your algebraically complicated answers.

(a) $f'\left(\dfrac{\pi}{12}\right)$, where $f(x) = \sec^2(2x) + \sin(4x)$.

$f'(x) = 2\sec(2x)\sec(2x)\tan(2x)2 + 4\cos(4x)$

$$f'\left(\frac{\pi}{12}\right) = 4\sec\left(\frac{2\pi}{12}\right)\sec\left(\frac{2\pi}{12}\right)\tan\left(\frac{2\pi}{12}\right) + 4\cos\left(\frac{4\pi}{12}\right) = 4\sec^2\left(\frac{\pi}{6}\right)\tan\left(\frac{\pi}{6}\right) + 4\cos\left(\frac{\pi}{3}\right)$$

$$= 4\cdot\left(\frac{2}{\sqrt{3}}\right)^2\cdot\frac{1}{\sqrt{3}} + 4\left(\frac{1}{2}\right) = \boxed{\frac{16}{3\sqrt{3}} + 2}$$

Note: $\sec x = \dfrac{1}{\cos x}$ and $\sec\dfrac{\pi}{6} = \dfrac{1}{\cos\frac{\pi}{6}} = \dfrac{1}{\left(\frac{\sqrt{3}}{2}\right)} = \dfrac{2}{\sqrt{3}}$

(b) $\dfrac{d}{dx}\ln\left(\dfrac{(x^2+1)^{\frac{3}{7}}\ e^{\tan x}}{\sqrt{1+\cos x}}\right) = \dfrac{d}{dx}\left[\ln\left((x^2+1)^{\frac{3}{7}}\right) + \ln e^{\tan x} - \ln\sqrt{1+\cos x}\right]$

$$= \frac{d}{dx}\left[\frac{3}{7}\ln(x^2+1) + \tan x - \frac{1}{2}\ln(1+\cos x)\right] = \frac{3}{7}\left(\frac{1}{x^2+1}\right)\cdot 2x + \sec^2 x - \frac{1}{2}\left(\frac{1}{1+\cos x}\right)\cdot(-\sin x)$$

$$= \boxed{\frac{6x}{7(x^2+1)} + \sec^2 x + \frac{\sin x}{2(1+\cos x)}}$$

(c) $g'(x)$, where $g(x) = \sqrt{1+\cos^7\left(\dfrac{5}{x}\right)}$

$$g'(x) = \boxed{\frac{1}{2\sqrt{1+\cos^7\left(\frac{5}{x}\right)}}\cdot 7\cos^6\left(\frac{5}{x}\right)\cdot\left(-\sin\left(\frac{5}{x}\right)\right)\cdot\left(\frac{-5}{x^2}\right)}$$

(d) $\dfrac{dy}{dx}$, if $e^{xy^3} + \sin^3 x = \ln(xy) + \sin(e^9)$.

$\dfrac{d}{dx}\left(e^{xy^3} + \sin^3 x\right) = \dfrac{d}{dx}\left(\ln(xy) + \sin(e^9)\right)$ Implicit Differentiation

$$e^{xy^3}\left(x3y^2\frac{dy}{dx} + y^3\right) + 3\sin^2 x\cos x = \frac{1}{xy}\left(x\frac{dy}{dx} + y\right) + 0$$

$$3xy^2e^{xy^3}\frac{dy}{dx} + y^3e^{xy^3} + 3\sin^2 x\cos x = \frac{1}{y}\frac{dy}{dx} + \frac{1}{x}$$

$$3xy^2e^{xy^3}\frac{dy}{dx} - \frac{1}{y}\frac{dy}{dx} = \frac{1}{x} - y^3e^{xy^3} - 3\sin^2 x\cos x$$

$$\left(3xy^2e^{xy^3} - \frac{1}{y}\right)\frac{dy}{dx} = \frac{1}{x} - y^3e^{xy^3} - 3\sin^2 x\cos x$$

$$\frac{dy}{dx} = \boxed{\frac{\frac{1}{x} - y^3e^{xy^3} - 3\sin^2 x\cos x}{3xy^2e^{xy^3} - \frac{1}{y}}}$$

(e) $g''(x)$, where $g(x) = \int_x^{2011} \sqrt{\ln t} + \ln\sqrt{t}\, dt$.

$$g'(x) = \frac{d}{dx}\int_x^{2011} \sqrt{\ln t} + \ln\sqrt{t}\, dt = -\frac{d}{dx}\int_{2011}^{x} \sqrt{\ln t} + \ln\sqrt{t}\, dt = -\left(\sqrt{\ln x} + \ln\sqrt{x}\right) \text{ ( FTC Part I )}$$

$$g''(x) = -\left(\frac{1}{2\sqrt{\ln x}}\left(\frac{1}{x}\right) + \frac{1}{\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\right)\right) = \boxed{-\left(\frac{1}{2x\sqrt{\ln x}} + \frac{1}{2x}\right)}$$

(f) $\frac{d}{dx}\, x^{\cos x}$

We can solve this two ways: first try Logarithmic Differentiation and using the properties of logs,

Let $y = x^{\cos x}$, so that $\ln y = \ln(x^{\cos x}) = \cos x \ln x$

Next use implicit differentation to differentiate both sides w.r.t $x$.

$\frac{d}{dx}(\ln y) = \frac{d}{dx}(\cos x \ln x)$

Then $\frac{1}{y}\frac{dy}{dx} = \cos x\left(\frac{1}{x}\right) + (\ln x)(-\sin x)$.

As a result, $\frac{dy}{dx} = y\left(\frac{\cos x}{x} - \sin x \ln x\right)$.

Finally, $\frac{dy}{dx} = \boxed{x^{\cos x}\left(\frac{\cos x}{x} - \sin x \ln x\right)}$.

The second option is to rewrite $y = x^{\cos x} = e^{\ln(x^{\cos x})} = e^{\cos x \ln x}$.

Then differentiate, $\frac{d}{dx}(x^{\cos x}) = \frac{d}{dx}\left(e^{\cos x \ln x}\right) = e^{\cos x \ln x}\left(\cos x\left(\frac{1}{x}\right) + \ln x(-\sin x)\right)$

$= \boxed{x^{\cos x}\left(\frac{\cos x}{x} - \ln x \sin x\right)}$.

**3.** [25 Points] Compute each of the following integrals.

(a) $$\int_{\frac{\pi}{18}}^{\frac{\pi}{9}} \tan(3x)\, dx = \int_{\frac{\pi}{18}}^{\frac{\pi}{9}} \frac{\sin(3x)}{\cos(3x)}\, dx = -\frac{1}{3}\int_{\frac{\sqrt{3}}{2}}^{\frac{1}{2}} \frac{1}{u}\, du = -\frac{1}{3}\ln|u|\Big|_{\frac{\sqrt{3}}{2}}^{\frac{1}{2}} = -\frac{1}{3}\left(\ln\left(\frac{1}{2}\right) - \ln\left(\frac{\sqrt{3}}{2}\right)\right)$$

$$= -\frac{1}{3}\left(\ln\left(\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)\right) = -\frac{1}{3}\left(\ln\left(\frac{1}{\sqrt{3}}\right)\right) = -\frac{1}{3}\left(\ln 1 - \ln\sqrt{3}\right) = -\frac{1}{3}\left(0 - \ln\sqrt{3}\right) = \boxed{\frac{\ln\sqrt{3}}{3}} \overset{\text{or}}{=} \boxed{\frac{\ln 3}{6}}$$

Here $\boxed{\begin{aligned} u &= \cos(3x) \\ du &= -3\sin(3x)dx \\ -\tfrac{1}{3}du &= \sin(3x)dx \end{aligned}}$ and $\boxed{\begin{aligned} x = \tfrac{\pi}{18} &\implies u = \cos\left(\tfrac{3\pi}{18}\right) = \cos\left(\tfrac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \\ x = \tfrac{\pi}{9} &\implies u = \cos\left(\tfrac{3\pi}{9}\right) = \cos\left(\tfrac{\pi}{3}\right) = \frac{1}{2} \end{aligned}}$

(b) $\displaystyle\int \frac{\left(x^{\frac{5}{2}}+1\right)^2}{x}\,dx = \int \frac{x^5+2x^{\frac{5}{2}}+1}{x}\,dx = \int x^4+2x^{\frac{3}{2}}+\frac{1}{x}\,dx = \boxed{\frac{x^5}{5}+\frac{4}{5}x^{\frac{5}{2}}+\ln|x|+C}$

(c) $\displaystyle\int_e^{e^4} \frac{3}{x\sqrt{\ln x}}\,dx = 3\int_1^4 \frac{1}{\sqrt{u}}\,du = 3\int_1^4 u^{-\frac{1}{2}}\,du = 6\sqrt{u}\Big|_1^4 = 6\left(\sqrt{4}-\sqrt{1}\right) = 6(2-1) = \boxed{6}$

Here $\boxed{\begin{array}{ll} u & = \ln x \\ du & = \frac{1}{x}\,dx \end{array}}$ and $\boxed{\begin{array}{ll} x = e & \implies u = \ln e = 1 \\ x = e^4 & \implies u = \ln e^4 = 4 \end{array}}$

(d) $\displaystyle\int e^{x^2+\ln x+1}\,dx = \int e^{x^2}e^{\ln x}e\,dx = e\int xe^{x^2}\,dx = \frac{e}{2}\int e^u\,du = \frac{e}{2}e^u+C = \boxed{\left(\frac{e}{2}\right)e^{x^2}+C}$

or $= \boxed{\dfrac{e^{x^2+1}}{2}+C}$

Here $\boxed{\begin{array}{ll} u & = x^2 \\ du & = 2x\,dx \\ \frac{1}{2}\,du & = x\,dx \end{array}}$

**4.** [10 Points] Give an $\varepsilon$–$\delta$ proof that $\displaystyle\lim_{x\to 2} 6-5x = -4$.

Scratchwork: we want $|f(x)-L| = |(6-5x)-(-4)| < \varepsilon$

$$|f(x)-L| = |(6-5x)-(-4)| = |6-5x+4| = |10-5x| = |-5(x-2)| = |-5||x-2| = 5|x-2|$$

(want $< \varepsilon$)

$$5|x-2| < \varepsilon \text{ means } |x-2| < \frac{\varepsilon}{5}$$

So choose $\delta = \frac{\varepsilon}{5}$ to restrict $0 < |x-2| < \delta$. That is $0 < |x-2| < \frac{\varepsilon}{5}$.

Proof: Let $\varepsilon > 0$ be given. Choose $\delta = \frac{\varepsilon}{5}$. Given $x$ such that $0 < |x-2| < \delta$, then as desired

$$|f(x)-L| = |(6-5x)-(-4)| = |-5x+10| = |-5(x-2)| = |-5||x-2| = 5|x-2| < 5\cdot\frac{\varepsilon}{5} = \varepsilon.$$

□

**5.** [10 Points] Let $f(x) = \dfrac{x+2}{x-3}$. Calculate $f'(x)$, using the **limit definition** of the derivative.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\frac{(x+h)+2}{(x+h)-3}-\frac{x+2}{x-3}}{h}$$

$$= \lim_{h\to 0}\frac{\left(\frac{(x+h+2)(x-3)-(x+2)(x+h-3)}{(x+h-3)(x-3)}\right)}{h}$$

$$= \lim_{h\to 0} \frac{x^2 + xh + 2x - 3x - 3h - 6 - (x^2 + xh - 3x + 2x + 2h - 6)}{h(x+h-3)(x-3)}$$

$$= \lim_{h\to 0} \frac{x^2 + xh + 2x - 3x - 3h - 6 - x^2 - xh + 3x - 2x - 2h + 6}{h(x+h-3)(x-3)}$$

$$= \lim_{h\to 0} \frac{-3h - 2h}{h(x+h-3)(x-3)} = \lim_{h\to 0} \frac{-5h}{h(x+h-3)(x-3)} = \lim_{h\to 0} \frac{-5}{(x+h-3)(x-3)}$$

$$= \boxed{\frac{-5}{(x-3)^2}}$$

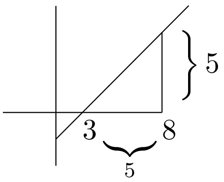
Free double check for yourself using the Quotient Rule:

$$f'(x) = \frac{(x-3)(1) - (x+2)(1)}{(x-3)^2} = \frac{x - 3 - x - 2}{(x-3)^2} = \frac{-5}{(x-3)^2} \qquad \text{Match!!}$$

**6.** [15 Points] Compute $\displaystyle\int_0^8 x - 3 \; dx$ using each of the following **three** different methods:

(*a*) Area interpretations of the definite integral,
(*b*) Fundamental Theorem of Calculus,
(*c*) Riemann Sums and the limit definition of the definite integral $***$.

***Recall $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\displaystyle\sum_{i=1}^{n} 1 = n$




(a) Area Above $x$-axis $= \frac{1}{2}(\text{base})\ (\text{height}) = \frac{1}{2}(5)(5) = \frac{25}{2}$

Area Below $x$-axis $= \frac{1}{2}(\text{base})\ (\text{height}) = \frac{1}{2}(3)(3) = \frac{9}{2}$

Then $\displaystyle\int_0^8 x - 3 \; dx = \frac{25}{2} - \frac{9}{2} = \frac{16}{2} = \boxed{8}$

(b) $\displaystyle\int_0^8 x - 3 \; dx = \frac{x^2}{2} - 3x \Big|_0^8 = \left(\frac{64}{2} - 24\right) - (0 - 0) = 32 - 24 = \boxed{8}$

(c) Here $a = 0, b = 8, \Delta x = \frac{8-0}{n} = \frac{8}{n}$ and $x_i = a + i\Delta x = 0 + \frac{8i}{n} = \frac{8i}{n}$.

$$
\begin{aligned}
\int_0^8 x - 3\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \ &= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{8i}{n}\right)\frac{8}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{8i}{n} - 3\right)\frac{8}{n} \\
&= \lim_{n\to\infty} \left(\frac{8}{n}\sum_{i=1}^{n}\frac{8i}{n} - \frac{8}{n}\sum_{i=1}^{n} 3\right) \\
&= \lim_{n\to\infty} \left(\frac{64}{n^2}\sum_{i=1}^{n} i - \frac{8}{n}(3n)\right) \\
&= \lim_{n\to\infty} \left(\frac{64}{n^2}\frac{n(n+1)}{2} - 24\right) \\
&= \lim_{n\to\infty} \left(\frac{64}{2}\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right) - 24\right) \\
&= \lim_{n\to\infty} \left(32(1)\left(1 + \frac{1}{n}\right) - 24\right) \\
&= 32 - 24 \\
&= \boxed{8}
\end{aligned}
$$

**7.** [10 Points] Find the equation of the tangent line to $y = \cos(\ln(x+1)) + \ln(\cos x) + e^{\sin x} + \sin(e^x - 1)$ at the point where $x = 0$.

$$y' = -\sin(\ln(x+1))\left(\frac{1}{x+1}\right) + \frac{1}{\cos x}(-\sin x) + e^{\sin x}\cos x + \cos(e^x - 1)e^x$$

$$y'(0) = -\sin(\ln(0+1))\left(\frac{1}{0+1}\right) + \frac{1}{\cos 0}(-\sin 0) + e^{\sin 0}\cos 0 + \cos(e^0 - 1)e^0$$

$= 0 + 0 + 1 + 1 = 2 \longleftarrow$ Slope

Point $(0, y(0)) = (0, 2)$

because $y(0) = \cos(\ln(0+1)) + \ln(\cos 0) + e^{\sin 0} + \sin(e^0 - 1) = \cos 0 + \ln 1 + e^0 + \sin 0$

$= 1 + 0 + 1 + 0 = 2$

Point-Slope Form

$y - 2 = 2(x - 0)$

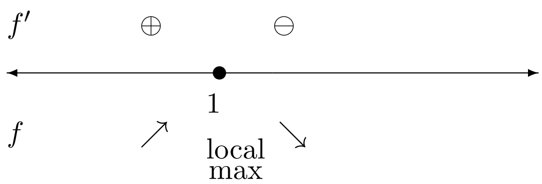
$\boxed{y = 2x + 2}$

**8.** [20 Points] Let $\boldsymbol{f(x) = \frac{x}{e^x} = xe^{-x}}$. For this function, discuss domain, vertical and horizontal asymptote(s), interval(s) of increase or decrease, local extreme value(s), concavity, and inflection point(s). Then use this information to present a detailed and labelled sketch of the curve. Take my word that $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = -\infty$.

- $f(x)$ has domain $(-\infty, \infty)$ so No Vertical Asymptotes.
- Vertical asymptotes: none
- Horizontal asymptotes: at $y = 0$ towards $\infty$, since $\lim_{x\to\infty} f(x) = 0$.
- First Derivative Information:

$f'(x) = xe^{-x}(-1) + e^{-x} = e^{-x}(-x + 1)$

The critical points occur where $f'$ is undefined (never here) or zero. Also note that the exponential function is always non-zero, which implies that $-x+1 = 0$ As a result, $x = 1$ is the critical number. Using sign testing/analysis for $f'$,



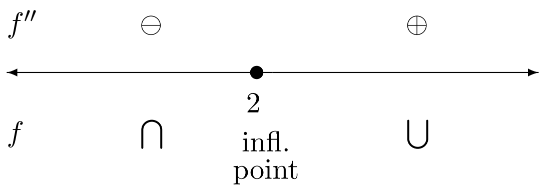

Therefore, $f$ is increasing on $(-\infty, 1)$ and decreasing on $(1, \infty)$ with local max at $(1, f(1)) = (1, e^{-1})$.

- Second Derivative Information

$f''(x) = e^{-x}(-1) + (-x + 1)e^{-x}(-1) = e^{-x}(-1 + x - 1) = e^{-x}(x - 2)$

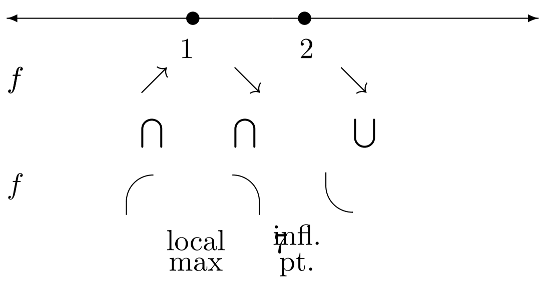
Possible inflection points occur when $f''$ is undefined (never here) or zero ($x = 2$) (again note the exponential piece is non-zero).

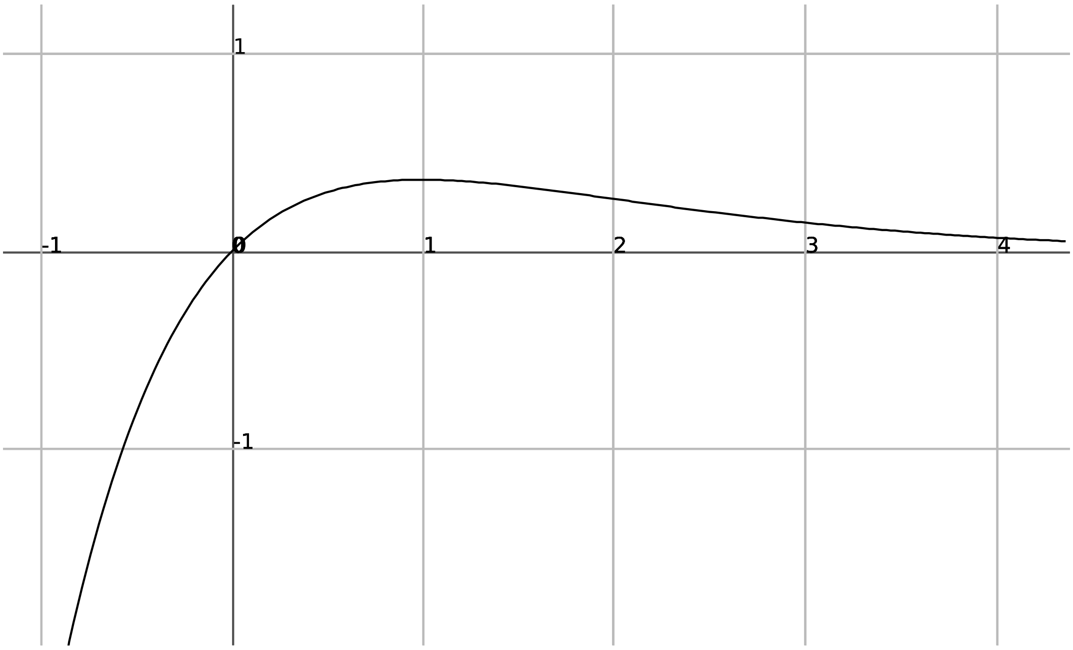
Using sign testing/analysis for $f''$,




Therefore, $f$ is concave down on $(-\infty, 2)$, whereas $f$ is concave up on $(2, \infty)$ with I.P. at $(2, f(2)) = (2, 2e^{-2})$.

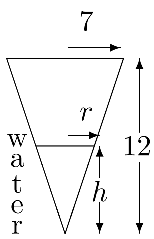
- Piece the first and second derivative information together






**9.** [15 Points] A conical tank, 14 feet across the entire top and 12 feet deep, is leaking water. The radius of the water level is decreasing at the rate of 2 feet per minute. How fast is the water leaking out of the tank when the radius of the water level is 2 feet? **Recall the volume of the cone is given by $V = \frac{1}{3}\pi r^2 h$

The cross section (with water level drawn in) looks like:




- Diagram
- Variables

Let $r$ = radius of the water level at time $t$
Let $h$ = height of the water level at time $t$
Let $V$ = volume of the water in the tank at time $t$
Find $\frac{dV}{dt}$ =? when $r = 2$ feet
and $\frac{dr}{dt} = -2\frac{\text{ft}}{\text{min}}$

- Equation relating the variables:

Volume= $V = \frac{1}{3}\pi r^2 h$

- Extra solvable information: Note that $h$ is not mentioned in the problem's info. But there is a relationship, via similar triangles, between $r$ and $h$. We must have

$$\frac{r}{7} = \frac{h}{12} \implies h = \frac{12r}{7}$$

After substituting into our previous equation, we get:

$V = \frac{1}{3}\pi r^2 \left(\frac{12r}{7}\right) = \frac{4}{7}\pi r^3$

• Differentiate both sides w.r.t. time $t$.

$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{4}{7}\pi r^3\right) \implies \frac{dV}{dt} = \frac{4}{7}\pi \cdot 3r^2 \cdot \frac{dr}{dt} \implies \frac{dV}{dt} = \frac{12}{7}\pi r^2 \frac{dr}{dt}$

• Substitute Key Moment Information (now and not before now!!!):

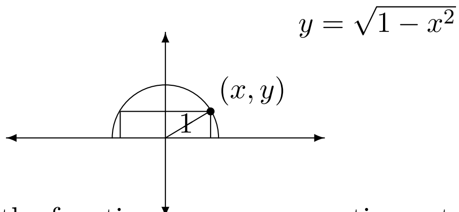
$\frac{dV}{dt} = \frac{12}{7}\pi(2)^2(-2)$

• Solve for the desired quantity:

$\boxed{\frac{dV}{dt} = -\frac{96\pi}{7}\frac{\text{ft}^3}{\text{min}}}$

• Answer the question that was asked: The water is leaking out of the tank at a rate of $\frac{96\pi}{7}$ cubic feet every minute.

**10.** [15 Points] Let $R$ be the region inside the top semicircle of radius one, centered at the origin, given by $y = \sqrt{1-x^2}$. Find the area of the largest rectangle that can be inscribed in this region $R$. Two vertices of the rectangle lie on the $x$–axis. Its other two vertices lie on the semicircle.




(Remember to state the domain of the function you are computing extreme values for.)

• Diagram: We already have a diagram.

• Variables:

Let $x = x$-coordinate of point $(x, y)$.
Let $y =$-coordinate of point $(x, y)$.
Let $A =$area of inscribed rectangle.

• Equation:

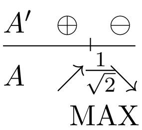
Then the area $A = 2xy = 2x\sqrt{1-x^2}$ must be maximized.

The (common-sense-bounds)domain of $A$ is $\boxed{\{x : 0 \leq x \leq 1\}}$.

• Maximize: Next $A' = (2x)\frac{(-2x)}{2\sqrt{1-x^2}} + \sqrt{1-x^2}(2) = \frac{-4x^2 + 4(1-x^2)}{2\sqrt{1-x^2}} = \frac{-8x^2+4}{2\sqrt{1-x^2}}$.

Setting $A' = 0$ we solve for $x^2 = \frac{1}{2}$ or $x = \frac{1}{\sqrt{2}}$. (We take the positive square root here because we're talking distance.)

Sign-testing the critical number does indeed yield a maximum for the area function.



• Answer: Since $x = \frac{1}{\sqrt{2}}$ then $y = \sqrt{1 - \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{1 - \frac{1}{2}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$. As a result, the largest area that occurs is $A = 2xy = 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = 1$ square unit.

**11.** [15 Points] Consider the region in the first quadrant bounded by $y = e^x + 1$, $y = 4$, and the $y$–axis.

(a) Draw a picture of the region. See me for a sketch.

Note that the curves intersect when $1 + e^x = 4$ which is when $e^x = 3$ which implies $x = \ln 3$.

(b) Compute the area of the region.

Area$= \displaystyle\int_0^{\ln 3} \text{top} - \text{bottom}\ dx = \int_0^{\ln 3} 4 - (e^x + 1)\ dx = \int_0^{\ln 3} 3 - e^x\ dx = 3x - e^x \Big|_0^{\ln 3}$

$= (3\ln 3 - e^{\ln 3}) - (0 - e^0) = 3\ln 3 - 3 + 1 = \boxed{\ln 27 - 2}$

(c) Compute the volume of the three-dimensional object obtained by rotating the region about the horizontal line $\boldsymbol{y = -2}$

Volume$= \displaystyle\int_0^{\ln 3} \pi[(\text{outer radius})^2 - (\text{inner radius})^2]\ dx = \int_0^{\ln 3} \pi[6^2 - (3 + e^x)^2]\ dx$

$= \displaystyle\int_0^{\ln 3} \pi[36 - (9 + 6e^x + e^{2x})]\ dx = \int_0^{\ln 3} \pi[27 - 6e^x - e^{2x}]\ dx = \pi[27x - 6e^x - \frac{1}{2}e^{2x}]\Big|_0^{\ln 3}$

$= \pi[(27\ln 3 - 6e^{\ln 3} - \frac{1}{2}e^{2\ln 3}) - (0 - 6e^0 - \frac{1}{2}e^0)] = \pi[27\ln 3 - 6(3) - \frac{1}{2}e^{\ln(3^2)} + 6 + \frac{1}{2}]$

$= \pi[27\ln 3 - 18 - \frac{9}{2} + 6 + \frac{1}{2}] = \pi[27\ln 3 - 12 - \frac{8}{2}] = \pi[27\ln 3 - 12 - 4] = \boxed{\pi[27\ln 3 - 16]}$

**12.** [15 Points] Consider an object moving on the number line such that its velocity at time $t$ seconds is $\boldsymbol{v(t) = 4 - t^2}$ feet per second. Also assume that the position of the object at one second is $\frac{5}{3}$.

(a) Compute the acceleration function $a(t)$ and the position function $s(t)$.

$a(t) = \boxed{-2t}$

$s(t) = \displaystyle\int 4 - t^2\ dt = 4t - \frac{t^3}{3} + C$

Use the initial condition $s(1) = \frac{5}{3}$

$s(1) = 4 - \frac{1}{3} + C \overset{\text{set}}{=} \frac{5}{3} \Rightarrow C = -2$

Finally, $s(t) = \boxed{4t - \frac{t^3}{3} - 2}$

(b) Compute the **total distance** travelled for $0 \le t \le 3$.

$$\begin{aligned}
\text{Total Distance} = \int_0^3 |4-t^2|\ dt &= \int_0^2 4-t^2\ dt + \int_2^3 -(4-t^2)\ dt \\
&= 4t - \frac{t^3}{3}\bigg|_0^2 + \left(-4t + \frac{t^3}{3}\right)\bigg|_2^3 \\
&= \left(8 - \frac{8}{3}\right) - (0-0) + (-12+9) - \left(-8 + \frac{8}{3}\right) \\
&= 8 - \frac{8}{3} - 3 + 8 - \frac{8}{3} \\
&= 13 - \frac{16}{3} \\
&= \frac{39}{3} - \frac{16}{3} \\
&= \boxed{\frac{23}{3}}
\end{aligned}$$