Amherst College, DEPARTMENT OF MATHEMATICS

Math 11, Final Examination, May 14, 2010

Answer Key

1. [20 Points] Evaluate each of the following limits. Please justify your answers. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist.

(a) $\lim\limits_{x\to 1} \dfrac{x^2-4x}{x^2+2x-8} = \dfrac{1-4}{1+2-8} = \dfrac{-3}{-5} = \boxed{\dfrac{3}{5}}$ by the Direct Substitution Property.

(b) $\lim\limits_{x\to 7} \dfrac{1}{|x-7|} = +\infty$ because RHL= LHL.

RHL: $\lim\limits_{x\to 7^+} \dfrac{1}{|x-7|} = \lim\limits_{x\to 7^+} \dfrac{1}{x-7} = \dfrac{1}{0^+} = +\infty$

LHL: $\lim\limits_{x\to 7^-} \dfrac{1}{|x-7|} = \lim\limits_{x\to 7^-} \dfrac{1}{-(x-7)} = \dfrac{1}{-0^-} = \dfrac{1}{0^+} = +\infty$

(c) $\lim\limits_{x\to 1^-} \dfrac{x^2+6x-7}{x^2-2x+1} = \lim\limits_{x\to 1^-} \dfrac{(x+7)(x-1)}{(x-1)(x-1)} = \lim\limits_{x\to 1^-} \dfrac{x+7}{x-1} = \dfrac{8}{0^-} = \boxed{-\infty}$

(d) $\lim\limits_{x\to -5} \dfrac{\dfrac{5}{x}-\dfrac{1}{x+4}}{x+5} = \lim\limits_{x\to -5} \dfrac{\dfrac{5(x+4)-x}{x(x+4)}}{x+5} = \lim\limits_{x\to -5} \dfrac{5x+20-x}{x(x+4)(x+5)} = \lim\limits_{x\to -5} \dfrac{4x+20}{x(x+4)(x+5)}$

$= \lim\limits_{x\to -5} \dfrac{4(x+5)}{x(x+4)(x+5)} = \lim\limits_{x\to -5} \dfrac{4}{x(x+4)} = \dfrac{4}{(-5)(-5+4)} = \boxed{\dfrac{4}{5}}$

2. [30 Points] Compute each of the following derivatives. Do not simplify your answers.

(a) $\dfrac{d}{dx}\left(\dfrac{e^{5x}-x^{\frac{3}{7}}}{\tan\sqrt{x}}\right) = \boxed{\dfrac{\tan\sqrt{x}\left(5e^{5x}-\frac{3}{7}x^{-\frac{4}{7}}\right)-\left(e^{5x}-x^{\frac{3}{7}}\right)\sec^2\sqrt{x}\cdot\dfrac{1}{2\sqrt{x}}}{(\tan\sqrt{x})^2}}$

(b) $\dfrac{dy}{dx}$, if $\sin^3 x + 5e^y = 7 + \ln(xy)$.

First we implicit differentiate both sides w.r.t. x. $\dfrac{d}{dx}\left(\sin^3 x + 5e^y\right) = \dfrac{d}{dx}\left(7+\ln(xy)\right)$.

Then $3\sin^2 x\cos x + 5e^y\dfrac{dy}{dx} = \dfrac{1}{xy}\left(x\dfrac{dy}{dx}+y\right) = \dfrac{1}{y}\dfrac{dy}{dx}+\dfrac{1}{x}$

As a result, $\left(5e^y-\dfrac{1}{y}\right)\dfrac{dy}{dx} = \dfrac{1}{x} - 3\sin^2 x\cos x$

Finally, $\boxed{\dfrac{dy}{dx} = \dfrac{\dfrac{1}{x} - 3\sin^2 x\cos x}{5e^y - \dfrac{1}{y}}}$.

(c) $\dfrac{d}{dx}\left(\displaystyle\int_x^7 \frac{e^t}{\ln t}\,dt\right) = -\dfrac{d}{dx}\left(\displaystyle\int_7^x \frac{e^t}{\ln t}\,dt\right) = \boxed{-\dfrac{e^x}{\ln x}}$ by FTC Part I

(d) $\dfrac{d}{dx}\left[\ln\left(\dfrac{\sqrt{x^7+1}\;e^{\sin x}}{(x^5+9)^3}\right)\right]$ (hint: you might want to simplify first)

$= \dfrac{d}{dx}\left[\ln\sqrt{x^7+1} + \ln e^{\sin x} - \ln((x^5+9)^3)\right] = \dfrac{d}{dx}\left[\ln((x^7+1)^{\frac{1}{2}} + \sin x - 3\ln(x^5+9)\right]$

$= \dfrac{d}{dx}\left[\dfrac{1}{2}\ln(x^7+1) + \sin x - 3\ln(x^5+9)\right] = \boxed{\tfrac{1}{2}\left(\dfrac{1}{x^7+1}\right)7x^6 + \cos x - 3\left(\dfrac{1}{x^5+9}\right)5x^4}$

(e) $f''(x)$, where $f(x) = e^{\cos x} + \dfrac{1}{x^7}$.

$f'(x) = e^{\cos x}(-\sin x) - 7x^{-8}$

$f''(x) = e^{\cos x}(-\cos x) - \sin x e^{\cos x}(-\sin x) + 56x^{-9} = \boxed{-e^{\cos x}\cos x + \sin^2 x e^{\cos x} + 56x^{-9}}$

(f) $f'(x)$, where $f(x) = x^x$.

We can solve this two ways: first try Logarithmic Differentiation and using the properties of logs,

Let $y = x^x$, so that $\ln y = \ln(x^x) = x\ln x$

Next use implicit differentation to differentiate both sides w.r.t x.

$\dfrac{d}{dx}(\ln y) = \dfrac{d}{dx}(x\ln x)$

Then $\dfrac{1}{y}\dfrac{dy}{dx} = x\left(\dfrac{1}{x}\right) + \ln x = 1 + \ln x$.

As a result, $\dfrac{dy}{dx} = y(1+\ln x)$.

Finally, $\dfrac{dy}{dx} = \boxed{x^x(1+\ln x)}$.

The second option is to rewrite $y = x^x = e^{\ln(x^x)} = e^{x\ln x}$.

Then differentiate, $\dfrac{d}{dx}(x^x) = \dfrac{d}{dx}\left(e^{x\ln x}\right) = e^{x\ln x}\left(x\left(\dfrac{1}{x}\right) + \ln x\right)$

$= e^{\ln(x^x)}(1+\ln x) = \boxed{x^x(1+\ln x)}$.

3. [25 Points] Compute each of the following integrals.

(a) $\displaystyle\int \frac{\left(x^{\frac{5}{2}}+1\right)^2}{x}\,dx = \int \frac{x^5+2x^{\frac{5}{2}}+1}{x}\,dx = \int x^4+2x^{\frac{3}{2}}+\frac{1}{x}\,dx = \frac{x^5}{5}+2\left(\frac{2}{5}\right)x^{\frac{5}{2}}+\ln|x|+C$

$\displaystyle = \boxed{\frac{x^5}{5}+\frac{4}{5}x^{\frac{5}{2}}+\ln|x|+C}$

(b) $\displaystyle\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \sin(2x)\cos(2x)\,dx = \frac{1}{2}\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} u\,du = \frac{1}{2}\cdot\frac{u^2}{2}\Big|_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} = \frac{1}{4}\left(\left(\frac{\sqrt{3}}{2}\right)^2-\left(\frac{1}{2}\right)^2\right) = \frac{1}{4}\left(\frac{3}{4}-\frac{1}{4}\right)$

$\displaystyle = \boxed{\frac{1}{8}}$

Here $\boxed{\begin{aligned} u &= \sin(2x) \\ du &= \cos(2x)\cdot 2dx \\ \tfrac{1}{2}du &= \cos(2x)dx \end{aligned}}$ and $\boxed{\begin{aligned} x = \tfrac{\pi}{12} \quad &\Rightarrow u = \sin\tfrac{\pi}{6} = \tfrac{1}{2} \\ x = \tfrac{\pi}{6} \quad &\Rightarrow u = \sin\tfrac{\pi}{3} = \tfrac{\sqrt{3}}{2} \end{aligned}}$

(c) $\displaystyle\int \frac{e^{5x}}{7+e^{5x}}\,dx = \frac{1}{5}\int \frac{1}{u}\,du = \frac{1}{5}\ln|u|+C = \boxed{\frac{1}{5}\ln|7+e^{5x}|+C}$

Here $\boxed{\begin{aligned} u &= 7+e^{5x} \\ du &= e^{5x}\cdot 5dx \\ \tfrac{1}{5}du &= e^{5x}dx \end{aligned}}$

(d) $\displaystyle\int \frac{e^{\frac{1}{x}}}{x^2}\,dx = -\int e^u\,du = -e^u+C = \boxed{-e^{\frac{1}{x}}+C}$

Here $\boxed{\begin{aligned} u &= \tfrac{1}{x} \\ du &= -\tfrac{1}{x^2}dx \\ -du &= \tfrac{1}{x^2}dx \end{aligned}}$

(e) $\displaystyle\int_e^{e^4} \frac{3}{x\sqrt{\ln x}}\,dx$

4. [10 Points] Give an ε-δ proof that $\displaystyle\lim_{x\to\,-1} 5x-4=-9$.

Scratchwork: we want $|f(x)-L| = |(5x-4)-(-9)| < \varepsilon$

$$|f(x)-L| = |(5x-4)-(-9)| = |5x-4+9| = |5x+5| = |5(x+1)| = |5||x+1| = 5|x-(-1)|$$

(want $< \varepsilon$)

$$5|x-(-1)| < \varepsilon \text{ means } |x-(-1)| < \frac{\varepsilon}{5}$$

So choose $\delta = \dfrac{\varepsilon}{5}$ to restrict $0 < |x - (-1)| < \delta$. That is $0 < |x - (-1)| < \dfrac{\varepsilon}{5}$.

Proof: Let $\varepsilon > 0$ be given. Choose $\delta = \dfrac{\varepsilon}{5}$. Given x such that $0 < |x - (-1)| < \delta$, then as desired

$$|f(x) - L| = |(5x - 4) - (-9)| = |5x + 5| = |5(x + 1)| = |5||x - (-1)| = 5|x - (-1)| < 5 \cdot \frac{\varepsilon}{5} = \varepsilon.$$

$\square$

5. [10 Points] Let $f(x) = \sqrt{5x - 2}$. Calculate $f'(x)$, using the **limit definition** of the derivative.

$$f'(x) = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{5(x + h) - 2} - \sqrt{5x - 2}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{5(x + h) - 2} - \sqrt{5x - 2}}{h} \cdot \frac{\sqrt{5(x + h) - 2} + \sqrt{5x - 2}}{\sqrt{5(x + h) - 2} + \sqrt{5x - 2}} = \lim_{h\to 0} \frac{(5(x + h) - 2) - (5x - 2)}{h(\sqrt{5(x + h) - 2} + \sqrt{5x - 2})}$$

$$= \lim_{h\to 0} \frac{5x + 5h - 2 - 5x + 2}{h(\sqrt{5(x + h) - 2} + \sqrt{5x - 2})} = \lim_{h\to 0} \frac{5h}{h(\sqrt{5(x + h) - 2} + \sqrt{5x - 2})}$$

$$= \lim_{h\to 0} \frac{5}{\sqrt{5(x + h) - 2} + \sqrt{5x - 2}} = \boxed{\frac{5}{2\sqrt{5x - 2}}}$$

Note: Can double check this by applying the Quotient Rule.

6. [10 Points] Suppose that $f(x) = \cos(e^x)$. Write the **equation** of the tangent line to the curve $y = f(x)$ when $x = \ln\left(\frac{\pi}{2}\right)$.

First the slope $f'(x) = -\sin(e^x) \cdot e^x$.

Then $f'\left((\ln\left(\frac{\pi}{2}\right)\right) = -\sin\left(e^{\ln\left(\frac{\pi}{2}\right)}\right) e^{\ln\left(\frac{\pi}{2}\right)} = -\sin\left(\frac{\pi}{2}\right) \cdot \left(\frac{\pi}{2}\right) = -(1) \cdot \left(\frac{\pi}{2}\right) = \boxed{-\frac{\pi}{2}}$.

Then the y-value is $f\left(\ln\left(\frac{\pi}{2}\right)\right) = \cos\left(e^{\ln\left(\frac{\pi}{2}\right)}\right) = \cos\frac{\pi}{2} = 0$

Therefore, the equation of the tangent line through the point $\left(\ln\left(\frac{\pi}{2}\right), 0\right)$ with slope $-\frac{\pi}{2}$, is

$y - 0 = -\frac{\pi}{2}\left(x - \ln\left(\frac{\pi}{2}\right)\right)$ or $\boxed{y = -\frac{\pi}{2}x + \frac{\pi}{2}\ln\left(\frac{\pi}{2}\right)}$.

7. [20 Points] Let $\boldsymbol{f(x) = xe^{-x}}$.

For this function, discuss domain, vertical and horizontal asymptote(s), interval(s) of increase or decrease, local extreme value(s), concavity, and inflection point(s). Then use this information to present a detailed and labelled sketch of the curve.

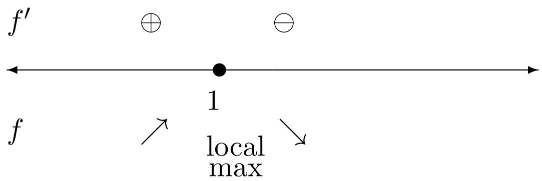
Take my word for it that

$$\boldsymbol{\lim_{x\to\infty} xe^{-x} = 0} \quad \text{and} \quad \boldsymbol{\lim_{x\to-\infty} xe^{-x} = -\infty}$$

- $f(x)$ has domain $(-\infty, \infty)$ so No Vertical Asymptotes.
- Vertical asymptotes: none
- Horizontal asymptotes: at $y = 0$ towards ∞, since $\lim_{x\to\infty} f(x) = 0$.
- First Derivative Information:

$f'(x) = xe^{-x}(-1) + e^{-x} = e^{-x}(-x + 1)$

The critical points occur where f' is undefined (never here) or zero. Also note that the exponential function is always non-zero, which implies that $-x+1 = 0$ As a result, $x = 1$ is the critical number. Using sign testing/analysis for f',



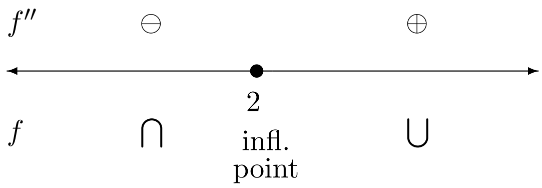
Therefore, f is increasing on $(-\infty, 1)$ and decreasing on $(1, \infty)$ with local max at $(1, f(1)) = (1, e^{-1})$.

- Second Derivative Information

$f''(x) = e^{-x}(-1) + (-x + 1)e^{-x}(-1) = e^{-x}(-1 + x - 1) = e^{-x}(x - 2)$

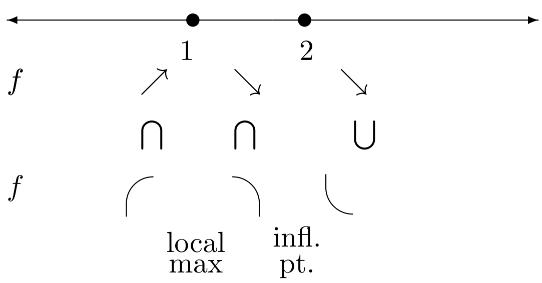
Possible inflection points occur when f'' is undefined (never here) or zero ($x = 2$) (again note the exponential piece is non-zero).

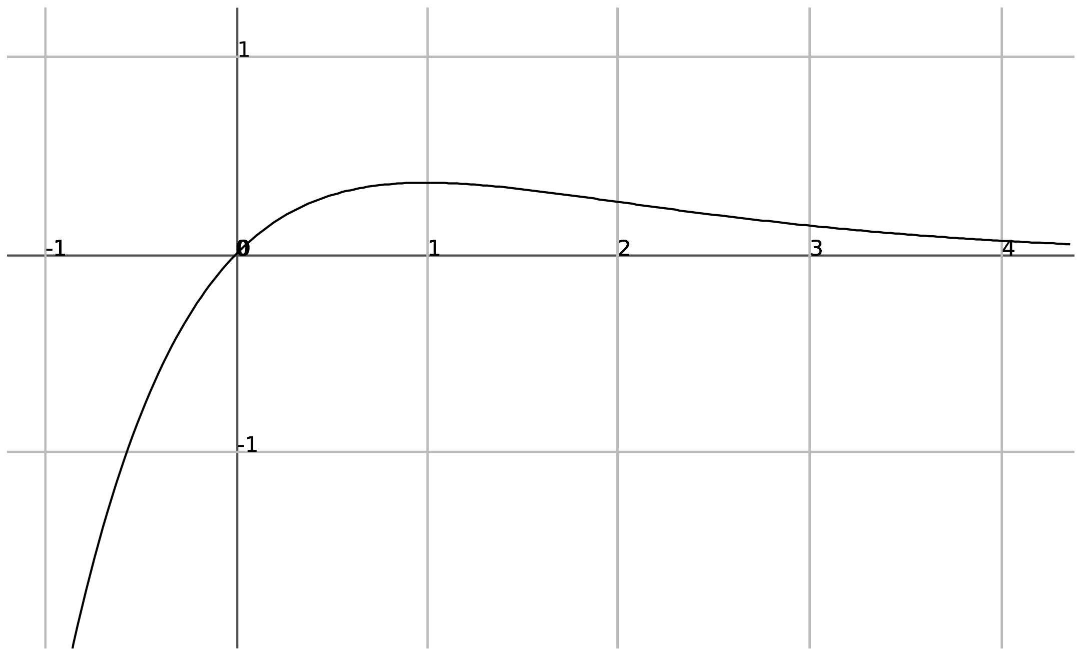
Using sign testing/analysis for f'',



Therefore, f is concave down on $(-\infty, 2)$, whereas f is concave up on $(2, \infty)$ with I.P. at $(2, f(2)) = (2, 2e^{-2})$.

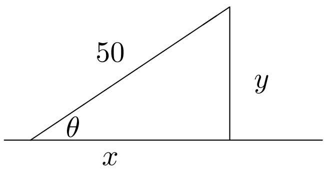
- Piece the first and second derivative information together





8. [15 Points] Suppose the top of a 50 foot ladder is sliding down a vertical wall at a rate of 10 feet per second. Consider the angle formed between the ground and the base of the ladder. At what rate is this angle changing when the top of the ladder is 30 feet above the ground?

• Diagram



• Variables

Let $x =$ distance between bottom of ladder and wall at time t
Let $y =$ distance between top of ladder and ground at time t
Let $\theta =$ angle formed by the ground and base of ladder at time t
Find $\dfrac{d\theta}{dt} =?$ when $y = 30$ ft
and $\dfrac{dy}{dt} = -10\dfrac{\text{ft}}{\text{sec}}$

• Equation relating the variables:

We have $\sin\theta = \dfrac{y}{50}$.

• Differentiate both sides w.r.t. time t.

$\dfrac{d}{dt}(\sin\theta) = \dfrac{d}{dt}\left(\dfrac{y}{50}\right) \implies \cos\theta\dfrac{d\theta}{dt} = \dfrac{1}{50}\dfrac{dy}{dt}$(Related Rates!)

• Substitute Key Moment Information (now and not before now!!!):

We're not given θ for this problem, but we can still compute $\cos\theta$ from trig. relations on the diagram's triangle with $\cos\theta = \frac{\text{adj}}{\text{hyp}}$. When $y = 30$, we can use the Pyth. Theorem to compute

$x = \sqrt{(50)^2 - (30)^2} = 40$. Finally, $\cos\theta = \dfrac{40}{50} = \dfrac{4}{5}$.

$\dfrac{4}{5}\dfrac{d\theta}{dt} = \dfrac{1}{50}(-10)$

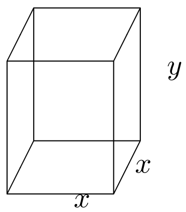
• Solve for the desired quantity:

$\dfrac{d\theta}{dt} = -\dfrac{10}{40} = -\dfrac{1}{4}\dfrac{\text{rad}}{\text{sec}}$

• Answer the question that was asked: The angle is decreasing at a rate of $\dfrac{1}{4}$ radians every second.

9. [15 Points] A box with a square base and a (flat) top is to be made to hold a volume of 27 cubic feet. Determine the **dimensions** that minimize the amount of material used.

(Remember to state the domain of the function you are computing extreme values for.)



The Volume of this box, which is fixed, is given as $V = x^2 y = 27 \Longrightarrow y = \dfrac{27}{x^2}$.

Then the Amount of Material must be minimized:

$$\begin{aligned} \text{M} &= \text{material for base} + \text{material for top} + \text{material for 4 sides} \\ &= x^2 + x^2 + 4xy \\ &= 2x^2 + 4xy \\ &= 2x^2 + 4x\left(\frac{27}{x^2}\right) \\ &= 2x^2 + \frac{108}{x} \end{aligned}$$

The (common-sense-bounds)domain of M are $\{x : x > 0\}$.

Next $M' = 4x - \dfrac{108}{x^2}$. Setting $M' = 0$ we solve for $x^3 = 27 \Rightarrow x = 3$ as the critical number.

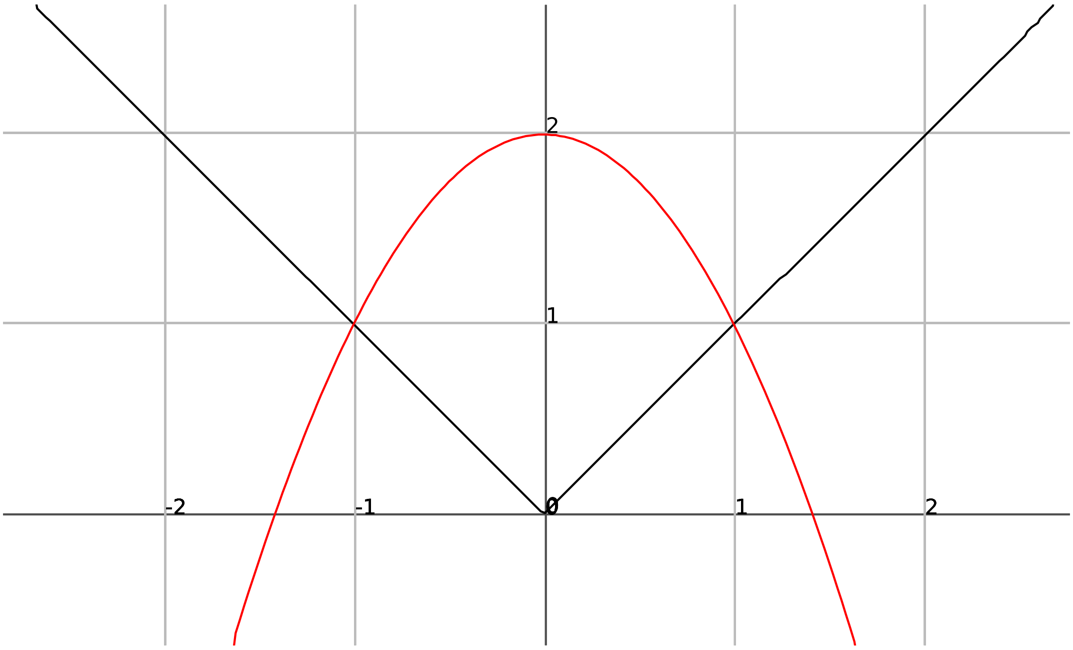
Sign-testing the critical number does indeed yield a minimum for the materials function.

$$\begin{array}{l|cc} \text{V}' & \ominus & \oplus \\ \hline \text{V} & \searrow \ 3 & \nearrow \\ & \text{MIN} & \end{array}$$

Since $x = 3$ then $y = \dfrac{27}{9} = 3$. As a result, the box of largest volume will measure $\boxed{\text{3x3x3}}$, each in feet $\rightarrow$ a cube.

10. [20 Points]

(a) Compute the **area** bounded by $y = |x|$, $y = 2 - x^2$. **Sketch** the region.

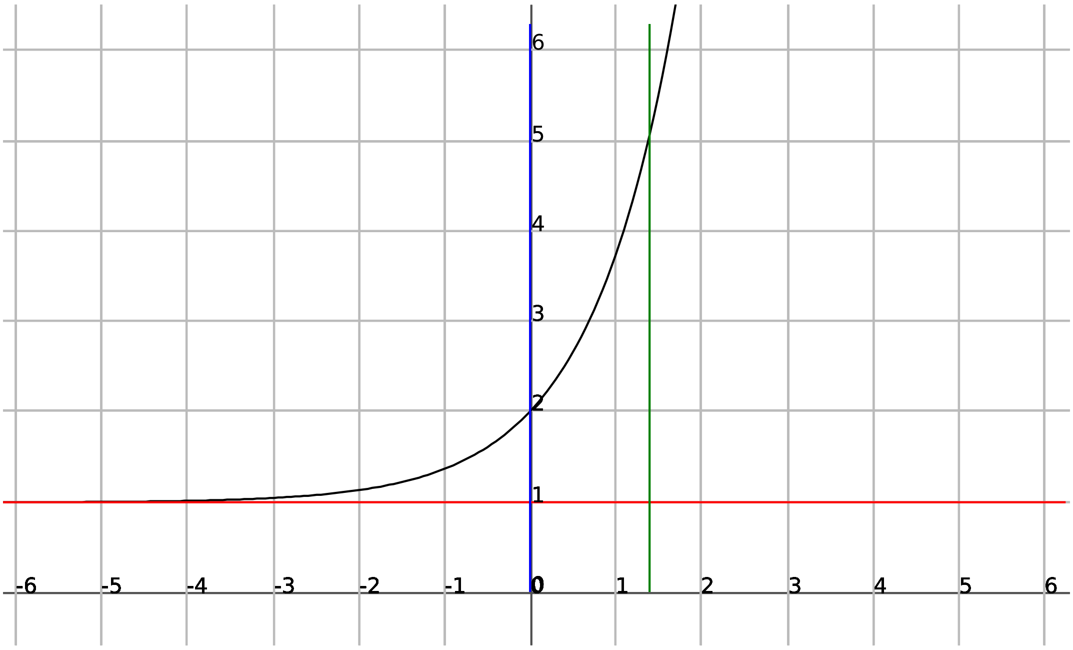


The curves intersect at $x = \pm 1$. We will integrate from $x = 0$ to $x = 1$, and double that area using symmetry.

$$\text{Area} = 2\int_0^1 \text{top} - \text{bottom}\ dx = 2\int_0^1 (2 - x^2) - x\ dx = 2\int_0^1 2 - x^2 - x\ dx = 2\left(2x - \frac{x^3}{3} - \frac{x^2}{2}\right)\Big|_0^1$$

$$= 2\left(\left(2 - \frac{1}{3} - \frac{1}{2}\right) - (0 - 0 - 0)\right) = 2\left(\frac{7}{6}\right) = \boxed{\frac{14}{6}}$$

(b) Consider the region in the plane bounded by $\boldsymbol{y = e^x + 1}$, $\boldsymbol{y = 1}$, $\boldsymbol{x = 0}$ and $\boldsymbol{x = \ln 4}$. Compute the **volume** of the 3-dimensional object obtained by rotating the region about the $\boldsymbol{x}$–axis. **Sketch** the region.



Volume$=\displaystyle\int_0^{\ln 4} \pi[(\text{outer radius})^2 - (\text{inner radius})^2]\ dx = \int_0^{\ln 4} \pi[(e^x+1)^2 - 1^2]\ dx$

$= \pi \displaystyle\int_0^{\ln 4} (e^{2x} + 2e^x + 1) - 1\ dx = \pi \int_0^{\ln 4} e^{2x} + 2e^x\ dx = \pi \left(\frac{1}{2}e^{2x} + 2e^x\right)\Big|_0^{\ln 4}$

$= \pi\left(\left(\frac{1}{2}e^{2\ln 4} + 2e^{\ln 4}\right) - \left(\frac{1}{2}e^0 + 2e^0\right)\right) = \pi\left(\left(\frac{1}{2}e^{\ln 16} + 2\cdot 4\right) - \left(\frac{1}{2}+2\right)\right)$

$= \pi\left(\left(\frac{1}{2}\cdot 16 + 8\right) - \frac{5}{2}\right) = \pi\left((8+8) - \frac{5}{2}\right) = \pi\left(16 - \frac{5}{2}\right) = \boxed{\dfrac{27\pi}{2}}$

11. [15 Points] Consider an object moving on the number line, starting at position 0, such that its acceleration at time t is $a(t) = 2$ feet per square second. Also assume that the object has initial velocity equaling -6 feet per second.

(a) Compute the velocity function $\boldsymbol{v(t)}$ and position function $\boldsymbol{s(t)}$.

$v(t) = 2t - v(0) = \boxed{2t - 6}$

$s(t) = t^2 - 6t + s(0) = \boxed{t^2 - 6t}$, because we know the initial condition $s(0) = 0$.

(b) Compute the **total distance** that it travels between time $t = 0$ and $t = 4$ seconds.

Therefore, Total Distance$=\displaystyle\int_0^4 |v(t)|\ dt = \int_0^4 |2t-6|\ dt = \int_0^3 -(2t-6)\ dt + \int_3^4 2t - 6\ dt$

$\displaystyle\int_0^3 -2t + 6\ dt + \int_3^4 2t - 6\ dt = -t^2 + 6t\Big|_0^3 + t^2 - 6t\Big|_3^4 = (-9+18) - (0-0) + (16-24) - (9-18)$

$= 9 - 8 + 9 = \boxed{10}$