Amherst College, DEPARTMENT OF MATHEMATICS

Math 11, Final Examination, December 19, 2009

• This examination booklet consists of 11 problems on 14 numbered pages. If you have received a defective copy, please notify your instructor immediately.

• This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, or other aids are permitted.

• You need *not* simplify algebraically complicated answers. However, numerical answers such as $\sin\left(\frac{\pi}{6}\right)$, $4^{\frac{3}{2}}$, $e^{\ln 4}$, $\ln(e^7)$, or $e^{3\ln 3}$ should be simplified.

• Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

1. [20 Points] Evaluate each of the following limits. Please justify your answers. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist.

(a) $\displaystyle\lim_{x\to 1} \frac{x^2+x-6}{x^2-6x+8}$

(b) $\displaystyle\lim_{x\to 3} \frac{3-x}{|x-3|}$

(c) $\displaystyle\lim_{x\to 1^+} \frac{x^2+x-2}{x^2-2x+1}$

(d) $\displaystyle\lim_{x\to -7} \frac{\frac{7}{x}-\frac{1}{x+6}}{x+7}$

2. [30 Points] Compute each of the following derivatives. Do not simplify your answers.

(a) $\displaystyle\frac{d}{dx}\left(\frac{x^3-\sin(3x)}{e^{4x}}\right)$

(b) $\displaystyle\frac{dy}{dx}$, if $x^2e^y = 1+\ln(xy)$.

(c) $\displaystyle\frac{d}{dx}\left(\int_x^2 \frac{\cos t}{3+\sin t}\,dt\right)$

(e) $\displaystyle\frac{d}{dx}\left[\ln\left(\frac{(x^2+5)^4\, e^{\tan x}}{\sqrt{x^3+2}}\right)\right]$

(f) $f''(x)$, where $f(x) = e^{\sin x} + \ln\sqrt{x}$.

(g) $f'(x)$, where $f(x) = x^{\sin x}$.

3. [25 Points] Compute each of the following integrals.

(a) $\displaystyle\int \frac{(1+\sqrt{x})^2}{x\sqrt{x}}\,dx$

(b) $\displaystyle\int_0^3 |x-1|\,dx$

(c) $\displaystyle\int \frac{e^x+\cos x}{e^x+\sin x}\,dx$

(d) $\int e^{6x}\sqrt{7+e^{6x}}\,dx$

(e) $\int_e^{e^4} \frac{1}{x\sqrt{\ln x}}\,dx$

4. [10 Points] Give an ε-δ proof that $\lim_{x\to 1} 8-3x=5$.

5. [10 Points] Let $f(x)=\sqrt{7x-3}$. Calculate $f'(x)$, using the **limit definition** of the derivative.

6. [10 Points] Suppose that $f(x)=\ln 2+\ln(\cos x)$. Write the **equation** of the tangent line to the curve $y=f(x)$ when $x=\frac{\pi}{3}$.

7. [20 Points] Sketch the graph of $y=f(x)$, where $f(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$.

This function is called the standard normal distribution, which is one of the most important models in statistics. Discuss any symmetry present in the graph. Clearly indicate horizontal asymptote(s), local minima/maxima, and inflection point(s) on the graph, as well as where the graph is increasing, decreasing, concave up and concave down. Take my word for it that

$$f'(x)=-\frac{x}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \qquad \text{and} \qquad f''(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}(x^2-1).$$

8. [15 Points] A train is travelling east on a straight track at 40 mph. The track is crossed by a road going north and south, and a house is on the road one mile south of the track. Draw the straight line connecting the house to the train. Consider the angle between the road and this line, as measured at the house. How fast is this angle changing when the train is 3 miles east of the road?

9. [20 Points] Let R be the region between $y=9-x^2$ and the x-axis. Find the area of the largest rectangle that can be inscribed in the region R. Two vertices of the rectangle lie on the x-axis. Its other two vertices above the x-axis lie on the parabola $y=9-x^2$.

(Remember to state the domain of the function you are computing extreme values for.)

10. [20 Points] Consider the region in the plane bounded by the curves $x=0$, $y=3$ and $y=1+e^x$.

(a) Draw a picture of the region.

(b) Compute the area of the region.

(c) Compute the volume of the 3-dimensional object obtained by rotating the region about the x-axis.

11. [20 Points] Consider an object moving on the number line such that its velocity at time t is $v(t)=\sin(t)+1$ ft/sec. Also assume that $s(0)=3$ ft, where as usual $s(t)$ is the position of the object at time t.

(a) Compute the acceleration $a(t)$ and position $s(t)$.

(b) Draw the graph of $v(t)$ for $0\le t\le 2\pi$ and explain why the object is always moving to the right.

(c) Compute the total distance travelled for $\pi/2\le t\le 2\pi$.