u-substitution technique for Integration

For this course we will have three main techniques of Integration.

1. *We know it* base Snap facts (with single variables)
2. Algebra, FOIL or *split-split* algebra
3. u-substitution

The technique of u-subsitution is a temporary convenience that essentially **reverses the Chain Rule**.

Example: The Chain Rule yields

$$\frac{d}{dx}\sin\left(x^3\right) = 3x^2\cos\left(x^3\right) \qquad \text{which gives } \int 3x^2\cos\left(x^3\right)\ dx = \sin\left(x^3\right) + C$$

.

Q: How can we compute these complicated integrals with *nested* pieces?

• The substitution method *hides a nested* part of your integrand and aims to match the derivative piece at about the same time.
• We need to choose u to be a nested chunk of your integrand, pretty much a grab-of-sorts of the inside portion of a composed function.
• Once you choose u as some hidden chunk of your integrand, that will yield a certain derivative du. In the end, we want to choose a substitution u that simplifies the Integral **and** also matches a part as the derivative.

$$\int f'(\underbrace{g(x)}_{u}) \cdot \underbrace{g'(x)\ dx}_{du} = \int f'(u)\ du = f(u) + C = f(g(x)) + C$$

$$\text{where } \boxed{\begin{aligned} u &= g(x) \\ du &= g'(x)\ dx \end{aligned}}$$

INDEFINITE Integrals: Always remember to add +C right away , as soon as you compute the Most General Antiderivative. The original variable always reappears when we re-substitute back for u.

$$\text{Ex: } \int \underbrace{\underbrace{x^6}\left(\underbrace{x^7 - 9}_{u}\right)^8 \underbrace{dx}}_{\frac{1}{7}du} = \frac{1}{7}\int u^8\ du = \frac{1}{7}\left(\frac{u^9}{9}\right) + C = \boxed{\frac{\left(x^7-9\right)^9}{63} + C}$$

$$\boxed{\begin{aligned} u &= x^7 - 9 \\ du &= 7x^6\ dx \\ \tfrac{1}{7}du &= x^6 dx \end{aligned}}$$

Ex: $\displaystyle\int \sin(\underbrace{6x}_{u})\ \underbrace{dx}_{\frac{1}{6}du} = \frac{1}{6}\int \sin u\ du = \frac{1}{6}(-\cos u) + C = \boxed{-\frac{1}{6}\cos(6x) + C}$

$$\boxed{\begin{aligned} u &= 6x \\ du &= 6\ dx \\ \tfrac{1}{6}du &= dx \end{aligned}}$$

Ex: $\displaystyle\int \overbrace{\frac{\overbrace{\sec^2 x}}{\sqrt{\underbrace{5+\tan x}_{u}}}\ \overbrace{dx}}^{du} = \int \frac{1}{\sqrt{u}}\ du = \int u^{-\frac{1}{2}}\ du = \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C = 2\sqrt{u} + C = \boxed{2\sqrt{5+\tan x} + C}$

$$\boxed{\begin{aligned} u &= 5+\tan x \\ du &= \sec^2 x\ dx \end{aligned}}$$

Ex:

$$\int \frac{5}{x^2\left(8+\frac{2}{x}\right)^3}\ dx = -\frac{5}{2}\int \frac{1}{u^3}\ du = -\frac{5}{2}\int u^{-3}\ du = -\frac{5}{2}\left(\frac{u^{-2}}{-2}\right) + C$$

$$= \frac{5}{4u^2} + C = \boxed{\frac{5}{4\left(8+\frac{2}{x}\right)^2} + C}$$

$$\boxed{\begin{aligned} u &= 8+\frac{2}{x} \\ du &= -\frac{2}{x^2}\ dx \\ -\tfrac{1}{2}du &= \frac{1}{x^2}\ dx \end{aligned}}$$

Ex:

$$\int \frac{7}{\sqrt{x}\left(3+\sqrt{x}\right)^2}\ dx = 7\int \frac{1}{\sqrt{x}\,(3+\sqrt{x})^2}\ dx = 14\int \frac{1}{u^2}\ du = 14\int u^{-2}\ du$$

$$= 14\left(\frac{u^{-1}}{-1}\right) + C = -\frac{14}{u} + C = \boxed{-\frac{14}{3+\sqrt{x}} + C}$$

$$\boxed{\begin{aligned} u &= 3+\sqrt{x} \\ du &= \tfrac{1}{2\sqrt{x}}\ dx \\ 2du &= \tfrac{1}{\sqrt{x}}\ dx \end{aligned}}$$

DEFINITE Integrals: Recall, you must change (or temporarily mark) your Limits of integration. The variables and Limits of Integration change *simultaneously*. Once you *switch* your Limits of Integration to u-values, then the original variable never reappears.

Ex:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x}{\cos^3 x}\,dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x}{(\cos x)^3}\,dx = -\int_{\frac{\sqrt{3}}{2}}^{\frac{1}{2}} \frac{1}{u^3}\,du = -\int_{\frac{\sqrt{3}}{2}}^{\frac{1}{2}} u^{-3}\,du = -\left(\frac{u^{-2}}{-2}\right)\Bigg|_{\frac{\sqrt{3}}{2}}^{\frac{1}{2}} = \frac{1}{2u^2}\Bigg|_{\frac{\sqrt{3}}{2}}^{\frac{1}{2}}$$

$$= \frac{1}{2\left(\frac{1}{2}\right)^2} - \frac{1}{2\left(\frac{\sqrt{3}}{2}\right)^2} = \frac{1}{2\left(\frac{1}{4}\right)} - \frac{1}{2\left(\frac{3}{4}\right)} = \frac{1}{\frac{1}{2}} - \frac{1}{\frac{3}{2}} = 2 - \frac{2}{3} = \frac{6}{3} - \frac{2}{3} = \boxed{\frac{4}{3}}$$

$$\boxed{\begin{aligned} u &= \cos x \\ du &= -\sin x\,dx \\ -du &= \sin x dx \end{aligned}} \text{ and } \boxed{\begin{aligned} x = \frac{\pi}{6} &\Rightarrow u = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \\ x = \frac{\pi}{3} &\Rightarrow u = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \end{aligned}}$$

OR here is an **ALTERNATE** option if you do not want to *Change* your Limits of Integration to u-limits. If you opt to *Mark* your Limits of Integration instead of *Changing* them to u Limits, then the original variable does reappear. Be careful not to mix and match x and u pieces.

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x}{\cos^3 x}\,dx = -\int_{x=\frac{\pi}{6}}^{x=\frac{\pi}{3}} \frac{1}{u^3}\,du = -\int_{x=\frac{\pi}{6}}^{x=\frac{\pi}{3}} u^{-3}\,du = -\left(\frac{u^{-2}}{-2}\right)\Bigg|_{x=\frac{\pi}{6}}^{x=\frac{\pi}{3}} = \frac{1}{2u^2}\Bigg|_{x=\frac{\pi}{6}}^{x=\frac{\pi}{3}}$$

$$= \frac{1}{2\cos^2 x}\Bigg|_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{1}{2\left(\cos\left(\frac{\pi}{3}\right)\right)^2} - \frac{1}{2\left(\cos\left(\frac{\pi}{6}\right)\right)^2} = \ldots = \boxed{\frac{4}{3}}$$

Note: Same u-sub as above, and same final values ...

Ex:

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{2}} \sin(3x)\,dx = \frac{1}{3}\int_{-\pi}^{\frac{3\pi}{2}} \sin u\,du = -\frac{1}{3}\cos u\Bigg|_{-\pi}^{\frac{3\pi}{2}}$$

$$= -\frac{1}{3}\cos\left(\frac{3\pi}{2}\right) - \left(-\frac{1}{3}\cos(-\pi)\right) = -0 + \frac{1}{3}(-1) = \boxed{-\frac{1}{3}}$$

$$\boxed{\begin{aligned} u &= 3x \\ du &= 3\,dx \\ \tfrac{1}{3}du &= dx \end{aligned}} \text{ and } \boxed{\begin{aligned} x = -\frac{\pi}{3} &\Rightarrow u = 3\left(-\frac{\pi}{3}\right) = -\pi \\ x = \frac{\pi}{2} &\Rightarrow u = 3\left(\frac{\pi}{2}\right) = \frac{3\pi}{2} \end{aligned}}$$

Here are some examples of *inverted* or *reverse* substitutions. When using a u-subsitution, we are fixing a (temporary) relationship between x and u for the entire problem. So, if there are any extra x variable leftover after the standard u-substitution, then you can solve the orginal choice of u in terms of x instead for x in terms of u. Then substitute that in for any leftover x's and then continue on with the antiderivative, etc.

Ex:

$$\int x\sqrt{x+1}\,dx = \int (u-1)\sqrt{u}\,du = \int u^{\frac{3}{2}} - u^{\frac{1}{2}}\,du = \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + C$$

$$= \boxed{\frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + C}$$

$$\boxed{\begin{aligned} u &= x+1 \Rightarrow x = u-1 \\ du &= 1\,dx \end{aligned}}$$

Ex:

$$\int \frac{x}{\sqrt{3-x}}\,dx = -\int \frac{3-u}{\sqrt{u}}\,du = -\int \frac{3}{\sqrt{u}} - \frac{u}{\sqrt{u}}\,du = -\int 3u^{-\frac{1}{2}} - u^{\frac{1}{2}}\,du$$

$$= -\left(3\left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right) - \frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right) + C = -6\sqrt{u} + \frac{2}{3}u^{\frac{3}{2}} + C$$

$$= \boxed{-6\sqrt{3-x} + \frac{2}{3}(3-x)^{\frac{3}{2}} + C}$$

$$\boxed{\begin{aligned} u &= 3-x \Rightarrow x = 3-u \\ du &= -1\,dx \\ -du &= dx \end{aligned}}$$