Math 106, Spring 2025

Homework #10

Due **Wednesday, March 5th** in Gradescope by 11:59 pm ET

Goal: Computing Area using the Limit Definition of the Definite Integral with Riemann Sums

Definition: the **Definite Integral** of a function f from $x = a$ to $x = b$ is given by

$$(\bullet) \quad \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty} [f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \ldots + f(x_i)\Delta x + \ldots + f(x_n)\Delta x]$$

Note: The Definite Integral is a Limit of a Sum of areas! Just think about this formula as

the **Limiting Value** *of the sum of the areas of finitely many (n) approximating rectangles.*

To compute definite integrals the *long (limit) way*, follow these steps:

Step 1: Given the integral $\int_a^b f(x)\,dx$, **pick off** or **identify** the **integrand** $f(x)$, and **limits of integration** (lower limit) a and (upper limit) b.

Step 2: Compute $\boxed{\Delta x = \frac{b-a}{n}}$. This Width of each partitioned interval will be in terms of n.

Step 3: Compute $\boxed{x_i = a + i\Delta x}$. Leave the i as your counter. You have the left-most endpoint a from Step 1. You have width Δx from Step 2. This endpoint x_i should be in terms of i and n.

Step 4: Plug x_i and Δx into the formula $(\bullet)$ above. Here it is again:

$$(\bullet) \quad \boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x} \quad \longleftarrow \textbf{MEMORIZE!}$$

Step 5: Use the following formulas for sum of integers i and finish evaluating the limit in n.

$$\sum_{i=1}^{n} 1 = n \qquad (*) \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad (**) \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$(***) \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Note: your final answer for the definite integral should be a **number** after you finish the limit.

Read through the entire next problem. Make sure you understand the formula to start, as well as the formulas for Δx and x_i. Because it doesn't feel natural yet, just trust the formulas right now. Use arrows to justify the final limiting size argument.

Evaluate $\displaystyle\int_0^6 x^2\, dx$ using the Limit Definition of the Definite Integral (and Riemann Sums).

Here $f(x) = x^2$, $a = 0$, $b = 6$, $\Delta x = \dfrac{b-a}{n} = \dfrac{6-0}{n} = \dfrac{6}{n}$ and $x_i = a + i\Delta x = 0 + i\left(\dfrac{6}{n}\right) = \dfrac{6i}{n}$.

$$\int_0^6 x^2 dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{6i}{n}\right)\left(\frac{6}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\left(\frac{6i}{n}\right)^2\right)\frac{6}{n}$$

$$= \lim_{n\to\infty} \frac{6}{n} \sum_{i=1}^{n} \frac{36i^2}{n^2} \quad \text{factor all non-}i\text{ pieces out}$$

$$= \lim_{n\to\infty} \left(\frac{216}{n^3}\sum_{i=1}^{n} i^2\right)$$

$$= \lim_{n\to\infty} \left(\frac{216}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right) \quad \text{using } (**)$$

$$= \lim_{n\to\infty} \left(\frac{216}{6}\cdot\frac{n(n+1)(2n+1)}{n^3}\right)$$

$$= \lim_{n\to\infty} \left(\frac{216}{6}\cdot\frac{n(n+1)(2n+1)}{n\cdot n\cdot n}\right) \quad \text{repartner}$$

$$= \lim_{n\to\infty} \left(\frac{216}{6}\cdot\left(\frac{\cancel{n}}{\cancel{n}}\right)\cdot\left(\frac{n+1}{n}\right)\cdot\left(\frac{2n+1}{n}\right)\right) \quad \text{split}$$

$$= \lim_{n\to\infty} \left(\frac{216}{6}\cdot 1\cdot\left(1+\overset{\nearrow 0}{\cancel{\tfrac{1}{n}}}\right)\cdot\left(2+\overset{\nearrow 0}{\cancel{\tfrac{1}{n}}}\right)\right)$$

$$= \frac{216}{6}\cdot 1\cdot 2 = \frac{216}{3} = \boxed{72}$$

Read through the entire next problem. Make sure you understand the formula to start, as well as the formulas for Δx and x_i. Here the lower limit of integration a is **not** 0.

Evaluate $\int_1^4 6 - 3x\ dx$ using the Limit Definition of the Definite Integral (and Riemann Sums).

Here $f(x) = 6 - 3x,\ a = 1,\ b = 4,\ \Delta x = \frac{b-a}{n} = \frac{4-1}{n} = \frac{3}{n}$

and $x_i = a + i\Delta x = 1 + i\left(\frac{3}{n}\right) = 1 + \frac{3i}{n}$.

$$
\begin{aligned}
\int_1^4 6 - 3x\ dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x &= \lim_{n\to\infty}\sum_{i=1}^{n} f\left(1+\frac{3i}{n}\right)\left(\frac{3}{n}\right)\\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\left(6-3\left(1+\frac{3i}{n}\right)\right)\frac{3}{n}\\
&= \lim_{n\to\infty}\left(\frac{3}{n}\sum_{i=1}^{n}\left(3-\frac{9i}{n}\right)\right) \quad \text{next distribute}\\
&= \lim_{n\to\infty}\left(\frac{3}{n}\left(\sum_{i=1}^{n}3-\sum_{i=1}^{n}\frac{9i}{n}\right)\right) \quad \text{split sums}\\
&= \lim_{n\to\infty}\left(\frac{3}{n}\sum_{i=1}^{n}3-\frac{3}{n}\sum_{i=1}^{n}\frac{9i}{n}\right)\\
&= \lim_{n\to\infty}\left(\frac{9}{n}\overset{n}{\cancel{\sum_{i=1}^{n}1}}-\frac{27}{n^2}\sum_{i=1}^{n}i\right) \quad \text{pull out non-}i\text{ pieces}\\
&= \lim_{n\to\infty}\frac{9}{\cancel{n}}(\cancel{n})-\frac{27}{n^2}\left(\frac{n(n+1)}{2}\right) \quad \text{using } (*)\\
&= \lim_{n\to\infty} 9-\frac{27}{2}\left(\frac{\cancel{n}}{\cancel{n}}\right)\left(\frac{n+1}{n}\right) \quad \text{repartner}\\
&= \lim_{n\to\infty} 9-\frac{27}{2}(1)\left(1+\overset{0}{\cancel{\frac{1}{n}}}\right) \quad \text{split}\\
&= 9-\frac{27}{2} = \frac{18}{2}-\frac{27}{2} = \boxed{-\frac{9}{2}}
\end{aligned}
$$

Read through the entire next problem. The lower limit of integration a is **negative** this time.

Evaluate $\displaystyle\int_{-2}^{3} x^2 - 4x + 3\ dx$ using the Limit Definition of the Definite Integral.

Here $f(x) = x^2 - 4x + 3, \quad a = -2, \quad b = 3, \quad \Delta x = \dfrac{b-a}{n} = \dfrac{3-(-2)}{n} = \dfrac{5}{n}$

and $x_i = a + x_i = -2 + i\left(\dfrac{5}{n}\right) = -2 + \dfrac{5i}{n}$.

$$
\begin{aligned}
\int_{-2}^{3} x^2 - 4x + 3\ dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(-2 + \frac{5i}{n}\right)\left(\frac{5}{n}\right) \\
&= \lim_{n\to\infty} \frac{5}{n} \sum_{i=1}^{n} \left(-2 + \frac{5i}{n}\right)^2 - 4\left(-2 + \frac{5i}{n}\right) + 3 \\
&= \lim_{n\to\infty} \frac{5}{n} \sum_{i=1}^{n} \left(4 - \frac{20i}{n} + \frac{25i^2}{n^2} + 8 - \frac{20i}{n} + 3\right) \quad \text{FOIL Algebra} \\
&= \lim_{n\to\infty} \frac{5}{n} \sum_{i=1}^{n} \left(\frac{25i^2}{n^2} - \frac{40i}{n} + 15\right) \quad \text{now distribute coeff/sum} \\
&= \lim_{n\to\infty} \frac{5}{n} \sum_{i=1}^{n} \frac{25i^2}{n^2} - \frac{5}{n} \sum_{i=1}^{n} \frac{40i}{n} + \frac{5}{n} \sum_{i=1}^{n} 15 \\
&= \lim_{n\to\infty} \frac{125}{n^3} \sum_{i=1}^{n} i^2 - \frac{200}{n^2} \sum_{i=1}^{n} i + \frac{75}{n} \cancelto{n}{\sum_{i=1}^{n} 1} \quad \text{now use (*)/(**)} \\
&= \lim_{n\to\infty} \frac{125}{n^3} \left(\frac{n(n+1)(2n+1)}{6}\right) - \frac{200}{n^2}\left(\frac{n(n+1)}{2}\right) + \frac{75}{\cancel{n}}(\cancel{n}) \\
&= \lim_{n\to\infty} \frac{125}{6}\left(\frac{\cancel{n}}{\cancel{n}}\right)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) - \frac{200}{2}\left(\frac{\cancel{n}}{\cancel{n}}\right)\left(\frac{n+1}{n}\right) + 75 \\
&= \lim_{n\to\infty} \frac{125}{6}(1)\left(1 + \cancelto{0}{\frac{1}{n}}\right)\left(2 + \cancelto{0}{\frac{1}{n}}\right) - (100)(1)\left(1 + \cancelto{0}{\frac{1}{n}}\right) + 75 \\
&= \frac{125}{6}(1)(1)(2) - (100)(1)(1) + 75 = \frac{125}{3} - 100 + 75 \\
&= \frac{125}{3} - 25 = \frac{125}{3} - \frac{75}{3} = \boxed{\frac{50}{3}}
\end{aligned}
$$

$\star$ Now complete these Homework problems:

1. Compute by hand, manually, the Area bounded above by the graph of $y = 2x+5$ and bounded below by $y = 0$ and between $x = 0$ and $x = 3$. Sketch the graph and shade the bounded region.

2. Evaluate $\int_0^3 2x+5\,dx$ using the Limit Definition of the Definite Integral (and Riemann Sums)

3. Compute by hand, manually, the **Net Area** bounded between the graph of $y = 4 - 2x$ and the x-axis ($y = 0$) and between $x = 1$ and $x = 5$. Sketch the graph and shade the bounded region.

4. Evaluate $\int_1^5 4-2x\,dx$ using the Limit Definition of the Definite Integral (and Riemann Sums)

NOTE: Recall the Definite Integral computes the Area bounded above the x-axis *minus* the Area bounded below the x-axis.

5. Evaluate $\int_0^4 x^2\,dx$ using the Limit Definition of the Definite Integral (and Riemann Sums). Sketch the graph and shade the bounded region.

6. Evaluate $\int_{-1}^2 x^2 - 3x + 2\,dx$ using the Limit Definition of the Definite Integral (and Riemann Sums). Sketch the graph and shade the bounded region.

<u>REGULAR OFFICE HOURS</u>

Monday: 12:00–3:00 pm

7:30–9:00 pm TA Andrew, SMUDD 207

Tuesday: 1:00–4:00 pm

Wednesday: 1:00-3:00 pm

Thursday: none for Professor

8:00–9:30 pm TA Andrew, SMUDD 208A

Friday: 12:00–2:00 pm

- Spring Break coming soon!