Exam 3 Spring 2018 Answer Key

1(a) $$y = \ln\left(\frac{\ln x \sqrt{1+e^x}}{(4-x^6)^3 e^{-\cos x}}\right)$$ simplify 1st using Log Properties

$$= \ln\left(\ln x \cdot \sqrt{1+e^x}\right) - \ln\left((4-x^6)^3 \cdot e^{-\cos x}\right)$$ $\ln\left(\frac{A}{B}\right) = \ln A - \ln B$

$$= \ln(\ln x) + \ln\left[(1+e^x)^{1/2}\right] - \left[\ln\left((4-x^6)^3\right) + \cancel{\ln e}^{-\cos x}\right]$$ $\ln(A \cdot B) = \ln A + \ln B$

$$= \ln(\ln x) + \frac{1}{2}\underbrace{\ln(1+e^x)}_{\text{does not simplify}} - 3\ln(4-x^6) + \cos x$$ $\ln(A^B) = B \cdot \ln A$

$$y' = \boxed{\frac{1}{\ln x} \cdot \frac{1}{x} + \frac{1}{2} \cdot \frac{1}{1+e^x} \cdot e^x - 3 \cdot \frac{1}{4-x^6} \cdot (-6x^5) - \sin x}$$

1(b) $$\frac{d}{dx}(\cos x)^{\sin x}$$

Let $y = (\cos x)^{\sin x}$

$$\ln y = \ln\left((\cos x)^{\sin x}\right)$$

$$\ln y = \sin x \cdot \ln(\cos x)$$

Differentiate

$$\frac{d}{dx}\ln y = \frac{d}{dx}\left(\sin x \cdot \ln(\cos x)\right)$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \sin x \cdot \frac{1}{\cos x} \cdot (-\sin x) + \ln(\cos x) \cdot \cos x$$

Solve $$\frac{dy}{dx} = \cancel{y}^{(\cos x)^{\sin x}}\left(-\frac{\sin^2 x}{\cos x} + \cos x \cdot \ln(\cos x)\right)$$

$$= \boxed{(\cos x)^{\sin x} \cdot \left(-\frac{\sin^2 x}{\cos x} + \cos x \cdot \ln(\cos x)\right)}$$

Resubstitute for y

recall: $\sin^2 x = (\sin x)^2 = \sin x \cdot \sin x$

1(c) $y = \ln\left(\ln\left(\cancel{\ln} e^x\right)\right) + \frac{e}{\ln x} + \frac{\ln x}{e} + \ln x \cdot e^x + \underbrace{\ln\left(x e^x\right)}_{\text{split}}$ simplify 1st ?

$$= \ln(\ln x) + \overset{\text{prep}}{e(\ln x)^{-1}} + \frac{1}{e}\cdot \ln x + \ln x \cdot e^x + \ln x + \underbrace{\cancel{\ln}(e^x)}_{x}$$

$$\boxed{y' = \frac{1}{\ln x}\cdot\frac{1}{x} - e(\ln x)^{-2}\cdot\frac{1}{x} + \frac{1}{e}\cdot\frac{1}{x} + \left(\ln x \cdot e^x + e^x \cdot \frac{1}{x}\right) + \frac{1}{x} + 1}$$

1(d) $y = \underbrace{\cancel{e}^{\cancel{\ln}(\ln x)}}_{\ln x} + \underbrace{\overbrace{\cancel{\ln}(e^5)}^{\text{constant}}}_{5} + \cancel{e}^{\cancel{\ln} x} + \underbrace{\overbrace{5^{\cancel{\ln e}}}^{\text{constant}}}_{5}$

$$= \ln x + 5 + x + 5$$

$$y' = \frac{1}{x} + 0 + 1 + 0 = \boxed{\frac{1}{x} + 1}$$

OR Note: If you don't Simplify 1st then y' looks as follows

$$y' = \cancel{e}^{\cancel{\ln}(\ln x)} \cdot \frac{1}{\ln x}\cdot\frac{1}{x} + \cancel{\frac{1}{e^5}}\cdot \cancel{e^5}\cdot 0 + \cancel{e}^{\cancel{\ln} x}\cdot\frac{1}{x} + 0$$

$$= \cancel{\ln x}\cdot\frac{1}{\cancel{\ln x}}\cdot\frac{1}{x} + 0 + \cancel{x}\cdot\frac{1}{\cancel{x}}$$

$$= \frac{1}{x} + 1$$ Match!

2(a) $f(x) = \frac{x+2}{e^x}$

factor off common factor

$$f'(x) = \frac{e^x(1) - (x+2)e^x}{(e^x)^2} = \frac{\cancel{e^x}(1 - x - 2)}{(e^x)^{\cancel{2}}} = \frac{-x-1}{e^x} \overset{\text{set}}{=} 0 \Rightarrow -x-1=0$$

$x = -1$ critical #

e^x + denominator

Sign Testing into $f'(x)$

	-2	-1	0
f'	+		−
f	↗	Local MAX	↘

Local Min: None

Local Max Value of $f(-1) = \frac{1}{e^{-1}} = e$

occurs at $x = -1$.

2(b) $y = [\ln(x+4)]^2$

$$y' = 2\,\frac{\ln(x+4)}{x+4} \overset{\text{set}}{=} 0$$

$$2\ln(x+4) = 0$$

$$\ln(x+4) = 0$$

$$e^{\ln(x+4)} = e^{0} \quad 1$$

$$x+4 = 1 \longrightarrow x = -3$$

y-value @ $x=-3$ given by $y(-3) = [\ln(\cancel{-3+4})]^2 = (\cancel{\ln 1})^2 = 0$

Point: $(-3, y(-3)) = \boxed{(-3, 0)}$

Note: problem asked for Point

3(a)

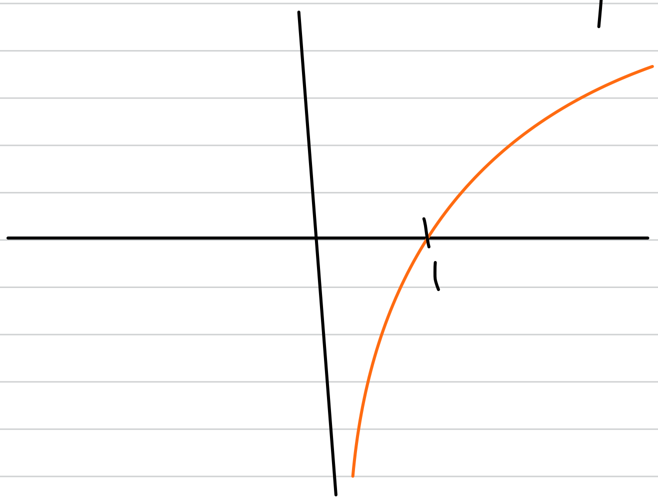
$y = e^x$

Domain $= \mathbb{R} = (-\infty, \infty)$

Range $= (0, \infty) = \{y : y > 0\}$

3(b)

$y = \ln x$



Domain $= (0, \infty) = \{x : x > 0\}$

Range $= \mathbb{R} = (-\infty, \infty)$

4(a) $$\int \frac{1}{x^3 e^{1/x^2}}\,dx = -\frac{1}{2}\int \frac{1}{e^u}\,du = -\frac{1}{2}\int e^{-u}\,du \overset{\text{K-rule}}{=} \cancel{-}\frac{1}{2}\cdot\frac{e^{-u}}{\cancel{-}1} + C$$

Right Awa

$$= \boxed{\frac{1}{2e^{1/x^2}} + C}$$

$$\boxed{\begin{aligned} u &= \frac{1}{x^2} \\ du &= -\frac{2}{x^3}\,dx \\ -\frac{1}{2}\,du &= \frac{1}{x^3}\,dx \end{aligned}}$$

$$4(b)\quad \int_1^{\sqrt6}\frac{x}{7-x^2}\,dx=-\frac12\int_6^1\frac1u\,du=-\frac12\ln|u|\Big|_6^1=-\frac12\left(\cancel{\ln 1}^{\,0}-\ln 6\right)$$

$$=\cancel{-}\frac12\left(\cancel{-}\ln 6\right)$$

$$=\boxed{\frac12\ln 6}\overset{\text{OR}}{=}\frac{\ln 6}{2}$$

$$\overset{\text{OR}}{=}\ln\sqrt6$$

$$\boxed{\begin{array}{l}u=7-x^2\\ du=-2x\,dx\\ -\frac12\,du=x\,dx\end{array}}\qquad\boxed{\begin{array}{l}x=1\Rightarrow u=7-1=6\\ x=\sqrt6\Rightarrow u=7-(\sqrt6)^2\\ \quad=7-6=1\end{array}}$$

$$4(c)\quad \int_{e^3}^{e^8}\frac{4}{x\sqrt{1+\ln x}}\,dx=\int_4^9\frac{4}{\sqrt u}\,du=4\frac{u^{1/2}}{1/2}\Big|_4^9=8\sqrt u\Big|_4^9$$

$u^{-1/2}$

$$=8\left(\cancel{\sqrt9}^{\,3}-\cancel{\sqrt4}^{\,2}\right)$$

$$=8(3-2)=\boxed{8}$$

$$\boxed{\begin{array}{l}u=1+\ln x\\ du=\frac1x\,dx\end{array}}\qquad\boxed{\begin{array}{l}x=e^3\Rightarrow u=1+\cancel{\ln e^3}=4\\ x=e^8\Rightarrow u=1+\cancel{\ln e^8}=9\end{array}}$$

split

$$4(d)\quad \int_{-3}^{-1}\frac{1-x}{x^2}\,dx=\int_{-3}^{-1}\frac1{x^2}-\frac{x}{x^2}\,dx=\int_{-3}^{-1}x^{-2}-\frac1x\,dx=\frac{x^{-1}}{-1}-\ln|x|\Big|_{-3}^{-1}$$

$$=-\frac1x-\ln|x|\Big|_{-3}^{-1}$$

Must have $|\cdot|$ Absolute Values

$$=\cancel{-}\frac{1}{\cancel{-}1}-\cancel{\ln|-1|}^{\,0}-\left(\cancel{-}\frac{1}{\cancel{-}3}-\ln|-3|\right)$$

$$=1-\frac13+\ln3=\boxed{\frac23+\ln 3}$$

$$4(e)\quad \int_{\pi/18}^{\pi/9}\tan(3x)\,dx=\int_{\pi/18}^{\pi/9}\frac{\sin(3x)}{\cos(3x)}\,dx=-\frac13\int_{\frac{\sqrt3}{2}}^{1/2}\frac1u\,du=-\frac13\ln|u|\Big|_{\frac{\sqrt3}{2}}^{\frac12}$$

$$=-\frac13\left(\ln\left(\frac12\right)-\ln\left(\frac{\sqrt3}{2}\right)\right)$$

$$=-\frac13\left(\cancel{\ln 1}^{\,0}-\cancel{\ln 2}-\ln\sqrt3+\cancel{\ln 2}\right)$$

$$=\cancel{-}\frac13\left(\cancel{-}\ln\sqrt3\right)$$

$$=\frac13\ln\left(3^{1/2}\right)=\frac16\cdot\ln3=\boxed{\frac{\ln 3}{6}}$$

Match!

$$\boxed{\begin{array}{l}u=\cos(3x)\\ du=-\sin(3x)\cdot3\,dx\\ -\frac13\,du=\sin(3x)\,dx\end{array}}$$

$$\boxed{\begin{array}{l}x=\frac{\pi}{18}\Rightarrow u=\cos\frac{\pi}{6}=\frac{\sqrt3}{2}\\ x=\frac{\pi}{9}\Rightarrow u=\cos\frac{\pi}{3}=\frac12\end{array}}$$

4(f) $\int_0^{\ln 2} \left(e^x + \frac{1}{e^{2x}}\right)^2 dx = \int_0^{\ln 2} \left(e^x + \frac{1}{e^{2x}}\right)\left(e^x + \frac{1}{e^{2x}}\right) dx$

FOIL

$$= \int_0^{\ln 2} e^{2x} + e^{-x} + e^{-x} + \frac{1}{e^{4x}}\, dx$$

K-rule

$\int e^{kx} dx = \frac{e^{kx}}{k} + C$

$$= \int_0^{\ln 2} e^{2x} + 2e^{-x} + e^{-4x}\, dx$$

$$= \frac{e^{2x}}{2} + \frac{2e^{-x}}{-1} + \frac{e^{-4x}}{-4} \Bigg|_0^{\ln 2}$$

$$= \frac{e^{2\ln 2}}{2} - 2e^{-\ln 2} - \frac{e^{-4\ln 2}}{4} - \left(\frac{\cancel{e^0}}{2} - 2\cancel{e^0} - \frac{1}{4}\cancel{e^0}\right)$$

$1 \quad 1 \quad 1$

$$= \frac{1}{2}\cancel{e}^{\ln(2^2)} - 2\cancel{e}^{\ln(2^{-1})} - \frac{1}{4}\cancel{e}^{\ln(2^{-4})} \qquad -\frac{1}{2} + 2 + \frac{1}{4}$$

$$= \frac{1}{\cancel{2}} \cdot \overset{2}{\cancel{4}} - \cancel{2} \cdot \frac{1}{\cancel{2}} - \frac{1}{4} \cdot \frac{1}{16} - \frac{1}{2} + 2 + \frac{1}{4}$$

$$= 2 - 1 - \frac{1}{64} - \frac{1}{2} + 2 + \frac{1}{4}$$

Match!

$$= 3 - \frac{1}{64} - \frac{1}{2} + \frac{1}{4} = \frac{192}{64} - \frac{1}{64} - \frac{32}{64} + \frac{16}{64} = \boxed{\frac{175}{64}}$$

$$\begin{array}{r} 1 \\ 64 \\ \times\ 3 \\ \hline 192 \end{array}$$

191 $\quad$ −16

5. $f(x) = \int f'(x)\,dx = \int \frac{1}{e^{2x}(1-2e^{-2x})^2}\,dx$

$$= \frac{1}{4}\int \frac{1}{u^2}\,du \quad u^{-2}$$

$$\boxed{\begin{aligned} u &= 1-2e^{-2x} \\ du &= 4e^{-2x}\,dx \\ \frac{1}{4}\,du &= \frac{1}{e^{2x}}\,dx \end{aligned}}$$

$$= \frac{1}{4}\left(\frac{u^{-1}}{-1}\right) + C$$

$$= \frac{-1}{4u} + C$$

$$= -\frac{1}{4(1-2e^{-2x})} + C$$

Test Initial Value to Solve for +C

$$f(0) = \frac{-1}{4(1-2\cancel{e^0})} + C \overset{\text{set}}{=} -1$$

$$\frac{\cancel{-}1}{4(\cancel{-}1)} + C = -1 \hookrightarrow \frac{1}{4} + C = -1 \hookrightarrow C = -\frac{5}{4}$$

Finally, $\boxed{f(x) = -\frac{1}{4(1-2e^{-2x})} - \frac{5}{4}}$

6. Prove $\frac{d}{dx}\ln x = \frac{1}{x}$

Let $y = \ln x$

Invert $e^y = \cancel{e}^{\cancel{\ln}x}$

$$e^y = x$$

Differentiate both Sides

$$\frac{d}{dx}(e^y) = \frac{d}{dx}(x)$$

$$e^y \cdot \frac{dy}{dx} = 1$$

Solve $\frac{dy}{dx} = \frac{1}{\cancel{e^y}\,x} = \frac{1}{x}$ ✓