Definite Integral Limit Definition using Riemann Sums

Definition: the **Definite Integral** of a function f from $x = a$ to $x = b$ is given by

$$(\bullet) \quad \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty} \left[f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \ldots + f(x_i)\Delta x + \ldots + f(x_n)\Delta x\right]$$

Note: The definite integral is a limit of a sum! Just think about this formula as

the **Limiting Value** *of the Sum of the Areas of finitely many (n) Approximating Rectangles.*

To compute Definite Integrals the *long (limit) way*, follow these steps:

Step 1: Given the Definite Integral $\int_a^b f(x)\,dx$, **pick off** or **identify** the **integrand** $f(x)$, and **limits of integration** a and b.

Step 2: Compute $\boxed{\Delta x = \dfrac{b-a}{n}}$. This width of each partitioned interval should be in terms of n.

Step 3: Compute $\boxed{x_i = a + i\Delta x}$. Leave the i as your counter. You have the left-most endpoint a from Step 1. You have width Δx from Step 2. This endpoint x_i should be in terms of i and n.

Step 4: Plug x_i and Δx into the formula $(\bullet)$ above. Here it is again:

$$(\bullet) \quad \boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x} \quad \longleftarrow \textbf{MEMORIZE!}$$

Step 5: Use the following formulas for sum of integers i and finish evaluating the limit in n.

$$\sum_{i=1}^{n} 1 = n$$

$$(*) \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$(**) \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$(***) \quad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Note: your final answer for the definite integral should be a **number** after you finish the limit.

1. **Read** through the entire next problem. Make sure you understand the formula to start, as well as the formulas for Δx and x_i. Because it doesn't feel natural yet, just trust the formulas right now.

Evaluate $\displaystyle\int_0^3 x^2\,dx$ using the Limit Definition of the Definite Integral with Riemann Sums.

Here $f(x) = x^2$, $a = 0$, $b = 3$, $\Delta x = \dfrac{b-a}{n} = \dfrac{3-0}{n} = \dfrac{3}{n}$ and $x_i = a + i\Delta x = 0 + i\left(\dfrac{3}{n}\right) = \dfrac{3i}{n}$.

$$\int_0^3 x^2 dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x \quad \lim_{n\to\infty}\sum_{i=1}^{n}\text{Heights}\cdot\text{Widths}$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\frac{3}{n}$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\left(\frac{3i}{n}\right)^2\right)\frac{3}{n} \quad \text{Evaluate Function}$$

$$= \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\frac{9i^2}{n^2}$$

$$= \lim_{n\to\infty}\frac{27}{n^3}\sum_{i=1}^{n} i^2 \quad \text{Pull out non-}i\text{ pieces from sum}$$

$$= \lim_{n\to\infty}\left(\frac{27}{n^3}\right)\left(\frac{n(n+1)(2n+1)}{6}\right) \text{ using } (**)$$

$$= \lim_{n\to\infty}\left(\frac{27}{6}\right)\left(\frac{n(n+1)(2n+1)}{n^3}\right)$$

$$= \lim_{n\to\infty}\left(\frac{27}{6}\right)\left(\frac{n(n+1)(2n+1)}{n\cdot n\cdot n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{27}{6}\right)\left(\frac{\cancel{n}}{\cancel{n}}\right)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) \quad \text{Repartner}$$

$$= \lim_{n\to\infty}\frac{27}{6}\cdot(1)\left(\cancelto{1}{1+\cancelto{0}{\frac{1}{n}}}\right)\left(\cancelto{2}{2+\cancelto{0}{\frac{1}{n}}}\right) \quad \text{Evaluate Limit}$$

$$= \frac{27}{6}(1)(2) = \frac{27}{3} = \boxed{9} \quad \text{Final Answer is a Value}$$

2. **Read** through the entire next problem. Make sure you understand the formula to start, as well as the formulas for Δx and x_i. Here the lower limit of integration a is **not** 0.

Evaluate $\displaystyle\int_1^5 9 - 2x \; dx$ using the Limit Definition of the Definite Integral with Riemann Sums.

Here $f(x) = 9 - 2x$, $a = 1$, $b = 5$, $\Delta x = \dfrac{b-a}{n} = \dfrac{5-1}{n} = \dfrac{4}{n}$
and $x_i = a + i\Delta x = 1 + i\left(\dfrac{4}{n}\right) = 1 + \dfrac{4i}{n}$.

$$\begin{aligned}
\int_1^5 9 - 2x \; dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \quad &= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(1 + \frac{4i}{n}\right)\frac{4}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(9 - 2\left(1 + \frac{4i}{n}\right)\right)\frac{4}{n} \quad \text{Evaluate Function} \\
&= \lim_{n\to\infty} \frac{4}{n} \sum_{i=1}^{n} \left(9 - 2 - \frac{8i}{n}\right) \\
&= \lim_{n\to\infty} \frac{4}{n} \sum_{i=1}^{n} \left(7 - \frac{8i}{n}\right) \\
&= \lim_{n\to\infty} \frac{4}{n} \left(\sum_{i=1}^{n} 7 - \sum_{i=1}^{n} \frac{8i}{n}\right) \quad \text{split sums} \\
&= \lim_{n\to\infty} \frac{28}{n} \cancelto{n}{\sum_{i=1}^{n} 1} - \frac{32}{n^2} \sum_{i=1}^{n} i \quad \text{Pull out non } i \text{ pieces} \\
&= \lim_{n\to\infty} \frac{28}{\cancel{n}}(\cancel{n}) - \frac{32}{n^2}\frac{n(n+1)}{2} \quad \text{using } (*) \\
&= \lim_{n\to\infty} 28 - \frac{32}{2}\left(\frac{\cancel{n}}{\cancel{n}}\right)\left(\frac{n+1}{n}\right) \quad \text{Repartner} \\
&= \lim_{n\to\infty} 28 - 16(1)\left(1 + \cancelto{0}{\frac{1}{n}}\right) \\
&= 28 - 16(1)(1) \\
&= \boxed{12}
\end{aligned}$$

3. Evaluate $\displaystyle\int_1^5 5-2x-x^2\,dx$ using the Limit Definition of the Definite Integral with Riemann Sums.

Here $f(x)=5-2x-x^2$, $\quad a=1$, $\quad b=5$, $\quad \Delta x=\dfrac{b-a}{n}=\dfrac{5-1}{n}=\dfrac{4}{n}$
and $x_i=a+i\Delta x=1+i\left(\dfrac{4}{n}\right)=1+\dfrac{4i}{n}$.

$$
\begin{aligned}
\int_1^5 5-2x-x^2\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x=\lim_{n\to\infty}\sum_{i=1}^{n} f\left(1+\frac{4i}{n}\right)\frac{4}{n}\\
&= \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n} 5-2\left(1+\frac{4i}{n}\right)-\left(1+\frac{4i}{n}\right)^2\\
&= \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n}\left(5-2-\frac{8i}{n}-1-\frac{8i}{n}-\frac{16i^2}{n^2}\right) \quad \text{FOIL Algebra}\\
&= \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n}\left(2-\frac{16i}{n}-\frac{16i^2}{n^2}\right) \quad \text{Combine like-terms}\\
&= \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n}2-\frac{4}{n}\sum_{i=1}^{n}\frac{16i}{n}-\frac{4}{n}\sum_{i=1}^{n}\frac{16i^2}{n^2}\\
&= \lim_{n\to\infty}\frac{8}{n}\cancelto{n}{\sum_{i=1}^{n}1}-\frac{64}{n^2}\sum_{i=1}^{n}i-\frac{64}{n^3}\sum_{i=1}^{n}i^2 \quad \text{Pull out non-}i\text{ pieces}\\
&= \lim_{n\to\infty}\frac{8}{\cancel{n}}(\cancel{n})-\frac{64}{n^2}\left(\frac{n(n+1)}{2}\right)-\frac{64}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)\\
&= \lim_{n\to\infty}8-\frac{64}{2}\left(\frac{\cancel{n}}{\cancel{n}}\right)\left(\frac{n+1}{n}\right)-\frac{64}{6}\left(\frac{\cancel{n}}{\cancel{n}}\right)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)\\
&= \lim_{n\to\infty}8-32\,(1)\left(1+\cancelto{0}{\frac{1}{n}}\right)-\frac{32}{3}\,(1)\left(1+\cancelto{0}{\frac{1}{n}}\right)\left(2+\cancelto{0}{\frac{1}{n}}\right)\\
&= 8-32(1)(1)-\frac{32}{3}(1)(1)(2)=-24-\frac{64}{3}=-\frac{72}{3}-\frac{64}{3}=\boxed{-\frac{136}{3}}
\end{aligned}
$$

Here the Lower Limit of Integration a is *Negative* this time. Careful with minus signs.

4. Evaluate $\displaystyle\int_{-3}^{5} x^2\,dx$ using the Limit Definition of the Definite Integral with Riemann Sums.

Here $f(x) = x^2$, $\quad a = -3$, $\quad b = 5$, $\quad \Delta x = \dfrac{b-a}{n} = \dfrac{5-(-3)}{n} = \dfrac{8}{n}$
and $x_i = a + i\Delta x = -3 + i\left(\dfrac{8}{n}\right) = -3 + \dfrac{8i}{n}$.

$$\begin{aligned}
\int_{-3}^{5} x^2\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(-3+\frac{8i}{n}\right)\left(\frac{8}{n}\right) \\
&= \lim_{n\to\infty} \frac{8}{n}\sum_{i=1}^{n} \left(-3+\frac{8i}{n}\right)^2 \\
&= \lim_{n\to\infty} \frac{8}{n}\sum_{i=1}^{n} 9 - \frac{48i}{n} + \frac{64i^2}{n^2} \quad \text{FOIL Algebra, Check signs carefully} \\
&= \lim_{n\to\infty} \frac{8}{n}\sum_{i=1}^{n} 9 - \frac{8}{n}\sum_{i=1}^{n}\frac{48i}{n} + \frac{8}{n}\sum_{i=1}^{n}\frac{64i^2}{n^2} \\
&= \lim_{n\to\infty} \frac{72}{n}\underbrace{\cancel{\sum_{i=1}^{n} 1}}_{n} - \frac{384}{n^2}\sum_{i=1}^{n} i + \frac{512}{n^2}\sum_{i=1}^{n} i^2 \quad \text{Pull out non-}i\text{ pieces} \\
&= \lim_{n\to\infty} \frac{72}{\cancel{n}}(\cancel{n}) - \frac{384}{n^2}\left(\frac{n(n+1)}{2}\right) + \frac{512}{n^2}\left(\frac{n(n+1)(2n+1)}{6}\right) \\
&= \lim_{n\to\infty} 72 - \frac{384}{2}\left(\frac{\cancel{n}}{\cancel{n}}\right)\left(\frac{n+1}{n}\right) + \frac{512}{6}\left(\frac{\cancel{n}}{\cancel{n}}\right)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) \quad \text{Repartner} \\
&= \lim_{n\to\infty} 72 - \frac{384}{2}(1)\left(1+\overbrace{\cancel{\frac{1}{n}}}^{0}\right) + \frac{512}{6}(1)\left(1+\overbrace{\cancel{\frac{1}{n}}}^{0}\right)\left(2+\overbrace{\cancel{\frac{1}{n}}}^{0}\right) \\
&= 72 - 192 + \frac{512}{3} = -120 + \frac{512}{3} = -\frac{360}{3} + \frac{512}{3} \\
&= \boxed{\frac{152}{3}}
\end{aligned}$$

5. Evaluate $\displaystyle\int_{-1}^{2} x^2 - 3x + 6\ dx$ using the Limit Definition of the Definite Integral with Riemann Sums.

Here $f(x) = x^2 - 3x + 6, \quad a = -1, \quad b = 2, \quad \Delta x = \dfrac{b-a}{n} = \dfrac{2-(-1)}{n} = \dfrac{3}{n}$
and $x_i = a + i\Delta x = -1 + i\left(\dfrac{3}{n}\right) = -1 + \dfrac{3i}{n}$.

$$
\begin{aligned}
\int_{-1}^{2} x^2 - 3x + 6\ dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(-1 + \frac{3i}{n}\right)\left(\frac{3}{n}\right) \\
&= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left(-1 + \frac{3i}{n}\right)^2 - 3\left(-1 + \frac{3i}{n}\right) + 6 \\
&= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left(1 - \frac{6i}{n} + \frac{9i^2}{n^2} + 3 - \frac{9i}{n} + 6\right) \\
&= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left(\frac{9i^2}{n^2} - \frac{15i}{n} + 10\right) \\
&= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \frac{9i^2}{n^2} - \frac{3}{n} \sum_{i=1}^{n} \frac{15i}{n} + \frac{3}{n} \sum_{i=1}^{n} 10 \\
&= \lim_{n\to\infty} \frac{27}{n^3} \sum_{i=1}^{n} i^2 - \frac{45}{n^2} \sum_{i=1}^{n} i + \frac{30}{n} \cancelto{n}{\sum_{i=1}^{n} 1} \\
&= \lim_{n\to\infty} \frac{27}{n^3} \left(\frac{n(n+1)(2n+1)}{6}\right) - \frac{45}{n^2}\left(\frac{n(n+1)}{2}\right) + \frac{30}{\cancel{n}}(\cancel{n}) \\
&= \lim_{n\to\infty} \frac{27}{6}\left(\frac{\cancel{n}}{\cancel{n}}\right)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) - \frac{45}{2}\left(\frac{\cancel{n}}{\cancel{n}}\right)\left(\frac{n+1}{n}\right) + 30 \\
&= \lim_{n\to\infty} \frac{27}{6}(1)\left(1 + \cancelto{0}{\frac{1}{n}}\right)\left(2 + \cancelto{0}{\frac{1}{n}}\right) - \frac{45}{2}(1)\left(1 + \cancelto{0}{\frac{1}{n}}\right) + 30 \\
&= \frac{27}{6}(1)(1)(2) - \frac{45}{2}(1)(1) + 30 = \frac{54}{6} - \frac{45}{2} + 30 \\
&= \frac{54}{6} - \frac{135}{6} + \frac{180}{6} = \frac{99}{6} = \boxed{\frac{33}{2}}
\end{aligned}
$$

6. Evaluate $\displaystyle\int_{-2}^{3} x^2 - 4x + 3\ dx$ using the Limit Definition of the Definite Integral with Riemann Sums.

Here $f(x) = x^2 - 4x + 3, \quad a = -2, \quad b = 3, \quad \Delta x = \dfrac{b-a}{n} = \dfrac{3-(-2)}{n} = \dfrac{5}{n}$
and $x_i = a + i\Delta x = -2 + i\left(\dfrac{5}{n}\right) = -2 + \dfrac{5i}{n}$.

$$\begin{aligned}
\int_{-2}^{3} x^2 - 4x + 3\ dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(-2 + \frac{5i}{n}\right)\left(\frac{5}{n}\right) \\
&= \lim_{n\to\infty} \frac{5}{n} \sum_{i=1}^{n} \left(-2 + \frac{5i}{n}\right)^2 - 4\left(-2 + \frac{5i}{n}\right) + 3 \\
&= \lim_{n\to\infty} \frac{5}{n} \sum_{i=1}^{n} \left(4 - \frac{20i}{n} + \frac{25i^2}{n^2} + 8 - \frac{20i}{n} + 3\right) \\
&= \lim_{n\to\infty} \frac{5}{n} \sum_{i=1}^{n} \left(\frac{25i^2}{n^2} - \frac{40i}{n} + 15\right) \\
&= \lim_{n\to\infty} \frac{5}{n} \sum_{i=1}^{n} \frac{25i^2}{n^2} - \frac{5}{n} \sum_{i=1}^{n} \frac{40i}{n} + \frac{5}{n} \sum_{i=1}^{n} 15 \\
&= \lim_{n\to\infty} \frac{125}{n^3} \sum_{i=1}^{n} i^2 - \frac{200}{n^2} \sum_{i=1}^{n} i + \frac{75}{n} \cancelto{n}{\sum_{i=1}^{n} 1} \\
&= \lim_{n\to\infty} \frac{125}{n^3} \left(\frac{n(n+1)(2n+1)}{6}\right) - \frac{200}{n^2}\left(\frac{n(n+1)}{2}\right) + \frac{75}{\cancel{n}}(\cancel{n}) \\
&= \lim_{n\to\infty} \frac{125}{6}\left(\frac{\cancel{n}}{\cancel{n}}\right)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) - \frac{200}{2}\left(\frac{\cancel{n}}{\cancel{n}}\right)\left(\frac{n+1}{n}\right) + 75 \\
&= \lim_{n\to\infty} \frac{125}{6}(1)\left(1 + \cancelto{0}{\frac{1}{n}}\right)\left(2 + \cancelto{0}{\frac{1}{n}}\right) - (100)(1)\left(1 + \cancelto{0}{\frac{1}{n}}\right) + 75 \\
&= \frac{125}{6}(1)(1)(2) - (100)(1)(1) + 75 = \frac{125}{3} - 100 + 75 \\
&= \frac{125}{3} - 100 + 75 = \frac{125}{3} - 25 = \frac{125}{3} - \frac{75}{3} = \boxed{\frac{50}{3}}
\end{aligned}$$