# Worksheet 5 Answer Key

Updated Spring 24

1 $\int_{-1}^{2} 2-3x-x^2\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\cdot\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(-1+\frac{3i}{n}\right)\cdot\left(\frac{3}{n}\right)$

$a=-1$, $b=2$ $\Big\rangle$ $\Delta x = \frac{b-a}{n} = \frac{2-(-1)}{n} = \frac{3}{n}$

$x_i = a + i\Delta x = -1 + i\left(\frac{3}{n}\right) = -1+\frac{3i}{n}$

$$= \lim_{n\to\infty} \frac{3}{n}\sum_{i=1}^{n} 2-3\left(-1+\frac{3i}{n}\right)-\left(-1+\frac{3i}{n}\right)^2$$

$$= \lim_{n\to\infty} \frac{3}{n}\sum_{i=1}^{n} 2+3-\frac{9i}{n}-\left(1-\frac{6i}{n}+\frac{9i^2}{n^2}\right)$$

$$= \lim_{n\to\infty} \frac{3}{n}\sum_{i=1}^{n} 2+3-\frac{9i}{n}-1+\frac{6i}{n}-\frac{9i^2}{n^2}$$

$$= \lim_{n\to\infty} \frac{3}{n}\sum_{i=1}^{n} 4-\frac{3i}{n}-\frac{9i^2}{n^2}$$

pull all non-$i$ values out of Series Sums

$$= \lim_{n\to\infty} \frac{3}{n}\sum_{i=1}^{n} 4 - \frac{3}{n}\sum_{i=1}^{n}\frac{3i}{n} - \frac{3}{n}\sum_{i=1}^{n}\frac{9i^2}{n^2}$$

$$= \lim_{n\to\infty} \frac{12}{n}\sum_{i=1}^{n} 1 - \frac{9}{n^2}\sum_{i=1}^{n} i - \frac{27}{n^3}\sum_{i=1}^{n} i^2$$

$n$

$i$-Formulas

repartner

$$= \lim_{n\to\infty} \frac{12}{\cancel{n}}\cdot\cancel{n} - \frac{9}{n^2}\left(\frac{n(n+1)}{2}\right) - \frac{27}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$$

$$= \lim_{n\to\infty} 12 - \frac{9}{2}\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right) - \frac{27}{6}\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)$$

$$= \lim_{n\to\infty} 12 - \frac{9}{2}(1)\left(\frac{n}{n}+\frac{1}{n}\right) - \frac{27}{6}(1)\left(\frac{n}{n}+\frac{1}{n}\right)\left(\frac{2n}{n}+\frac{1}{n}\right)$$

$$= \lim_{n\to\infty} 12 - \frac{9}{2}\left(1+\frac{1}{n}\right) - \frac{27}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

Finally $\rightsquigarrow$ Let $n\to\infty$

$$= 12-\frac{9}{2}-\frac{27}{6}(1)(2) = 12-\frac{9}{2}-\frac{27}{3}$$

$$= 12-\frac{9}{2}-9 = 3-\frac{9}{2} = \frac{6}{2}-\frac{9}{2} = -\frac{3}{2}$$

Match!

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Optional FTC $\int_{-1}^{2} 2-3x-x^2\,dx = 2x-\frac{3x^2}{2}-\frac{x^3}{3}\Big|_{-1}^{2} = 4-\frac{12}{2}-\frac{8}{3}-\left(-2-\frac{3}{2}+\frac{1}{3}\right)$

$$= \cancel{4}-\cancel{6}-\frac{8}{3}+\cancel{2}+\frac{3}{2}-\frac{1}{3} = \frac{3}{2}-\frac{9}{3} = \frac{3}{2}-3 = \frac{3}{2}-\frac{6}{2} = \frac{-3}{2}$$

Cancel!

Match!

Now we can use "Quick Way" which is the Fundamental Theorem of Calculus.

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$ where $F(x)$ is any Antiderivative of $f$.

means $F'(x) = f(x)$

2. $$\int_0^{\pi/3} \sec^2\theta \, d\theta = \tan\theta \Big|_0^{\pi/3} = \cancel{\tan\frac{\pi}{3}}^{\sqrt{3}} - \cancel{\tan 0}^{0} = \sqrt{3} - 0 = \sqrt{3}$$

3. $$\int_{-\pi}^{\frac{\pi}{3}} 7\cos x\,dx = 7\int_{-\pi}^{\frac{\pi}{3}} \cos x\,dx = 7\sin x\Big|_{-\pi}^{\frac{\pi}{3}} = 7\left(\cancel{\sin\frac{\pi}{3}}^{\frac{\sqrt{3}}{2}} - \cancel{\sin(-\pi)}^{0}\right)$$

$$= 7\left(\frac{\sqrt{3}}{2} - 0\right) = \frac{7\sqrt{3}}{2}$$

4. $$\int_{-2}^{-1} x - \frac{5}{x^3}\,dx = \int_{-2}^{-1} x - 5x^{-3}\,dx = \frac{x^2}{2} \cancel{-} \frac{5x^{-2}}{\cancel{-}2}\Big|_{-2}^{-1} = \frac{x^2}{2} + \frac{5}{2x^2}\Big|_{-2}^{-1}$$

$$= \frac{\cancel{(-1)^2}^{1}}{2} + \frac{5}{2\cancel{(-1)^2}_{1}} - \left(\frac{\cancel{(-2)^2}^{4\,2}}{2} + \frac{5}{2\underbrace{(-2)^2}_{4}}\right)$$

push negative powers to the denominator to evaluate

$$= \underbrace{\frac{1}{2} + \frac{5}{2}}_{6/2 = 3} - \left(2 + \frac{5}{8}\right) = \underbrace{3 - 2}_{1} - \frac{5}{8} = \frac{8}{8} - \frac{5}{8} = \frac{3}{8}$$

5. $$\int_0^{\pi/6} (\tan x + \sec x)\sec x\,dx = \int_0^{\pi/6} \sec x \cdot \tan x + \sec^2 x\,dx = \sec x + \tan x\Big|_0^{\pi/6}$$

$$= \cancel{\sec\frac{\pi}{6}}^{\frac{2}{\sqrt{3}}} + \cancel{\tan\frac{\pi}{6}}^{\frac{1}{\sqrt{3}}} - \left(\cancel{\sec 0}^{1} + \cancel{\tan 0}^{0}\right)$$

$$= \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} - 1 = \frac{3}{\sqrt{3}} - 1 = \sqrt{3} - 1$$

$$\sec\frac{\pi}{6} = \frac{1}{\cancel{\cos\frac{\pi}{6}}_{\frac{\sqrt{3}}{2}}} = \frac{2}{\sqrt{3}}$$

6. $\int_1^2 \left(x-\frac{1}{x}\right)^2 dx = \int_1^2 \left(x-\frac{1}{x}\right)\left(x-\frac{1}{x}\right) dx = \int_1^2 x^2 - \overset{-1}{\cancel{\frac{x}{x}}} - \overset{-1}{\cancel{\frac{x}{x}}} + \frac{1}{x^2}\, dx$

$$= \int_1^2 x^2 - 2 + x^{-2}\, dx = \frac{x^3}{3} - 2x + \frac{x^{-1}}{-1}\Bigg|_1^2 = \frac{x^3}{3} - 2x - \frac{1}{x}\Bigg|_1^2$$

$$= \frac{8}{3} - 4 - \frac{1}{2} - \left(\frac{1}{3} - 2 - 1\right) = \frac{8}{3} - 4 - \frac{1}{2} - \frac{1}{3} + 2 + 1$$

$$= \frac{7}{3} - \frac{1}{2} - 1 = \frac{14}{6} - \frac{3}{6} - \frac{6}{6} = \frac{5}{6}$$

7. split

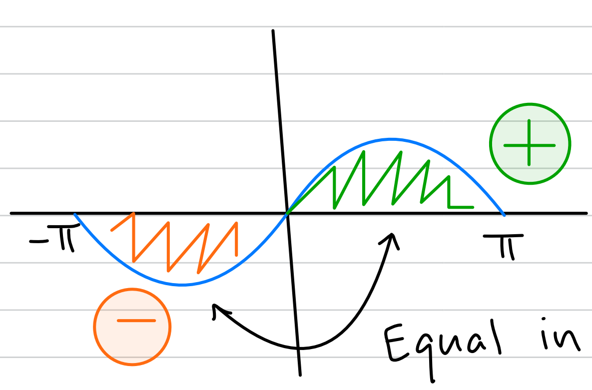
$\int_1^4 \frac{\sqrt{x} - x^2}{x} dx = \int_1^4 \frac{\sqrt{x}}{x} - \frac{x^2}{x}\, dx = \int_1^4 x^{-\frac{1}{2}} - x\, dx = \frac{x^{1/2}}{1/2} - \frac{x^2}{2}\Bigg|_1^4$

$$= 2\sqrt{x} - \frac{x^2}{2}\Bigg|_1^4 = 2\overset{2}{\cancel{\sqrt{4}}} - \overset{-8}{\cancel{\frac{16}{2}}} - \left(2\overset{}{\cancel{\sqrt{1}}} - \frac{1}{2}\right)$$

$$= 4 - 8 - 2 + \frac{1}{2} = -6 + \frac{1}{2} = -\frac{12}{2} + \frac{1}{2} = -\frac{11}{2}$$

8. $\int_{-\pi}^{\pi} \sin x\, dx = -\cos x \Bigg|_{-\pi}^{\pi} = -\overset{-1}{\cancel{\cos\pi}} + \overset{-1}{\cancel{\cos(-\pi)}} = 1 - 1 = 0$ Match

Makes sense because the Definite Integral computes the Area bounded Above x-axis MINUS Area bounded Below x-axis




Equal in Size by Symmetry ⇒ Cancel

FTC First

$$9(a)\quad \int_2^5 x^2\,dx = \frac{x^3}{3}\Big|_2^5 = \frac{5^3}{3} - \frac{2^3}{3} = \frac{125}{3} - \frac{8}{3} = \frac{117}{3} = 39$$

Limit Definition

$$9(b)\quad \int_2^5 x^2\,dx \;= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(2+\frac{3i}{n}\right)\cdot\left(\frac{3}{n}\right)$$

Here $f(x) = x^2$

$a = 2 \quad b = 5$

$\Delta x = \frac{b-a}{n} = \frac{5-2}{n} = \frac{3}{n}$

$x_i = a + i\Delta x$

$= 2 + i\left(\frac{3}{n}\right)$

$= 2 + \frac{3i}{n}$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} f\left(2+\frac{3i}{n}\right)$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left(2+\frac{3i}{n}\right)^2$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} 4 + \frac{12i}{n} + \frac{9i^2}{n^2}$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} 4 + \sum_{i=1}^{n} \frac{12i}{n} + \sum_{i=1}^{n} \frac{9i^2}{n^2}$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} 4 + \frac{3}{n} \sum_{i=1}^{n} \frac{12i}{n} + \frac{3}{n} \sum_{i=1}^{n} \frac{9i^2}{n^2}$$

$$= \lim_{n\to\infty} \frac{12}{n} \cancel{\sum_{i=1}^{n} 1}^{\,n} + \frac{36}{n^2} \cancel{\sum_{i=1}^{n} i} + \frac{27}{n^3} \cancel{\sum_{i=1}^{n} i^2}$$

$$= \lim_{n\to\infty} \frac{12}{\cancel{n}} \cdot \cancel{n} + \frac{36}{n^2}\left(\frac{n(n+1)}{2}\right) + \frac{27}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$$

$$= \lim_{n\to\infty} 12 + \frac{\cancel{36}^{18}}{\cancel{2}}\left(\cancel{\frac{n}{n}}^{1}\right)\left(\frac{n+1}{n}\right) + \frac{27}{6}\left(\cancel{\frac{n}{n}}\right)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)$$

$$= \lim_{n\to\infty} 12 + 18(1)\cdot\left(\cancel{1+\cancel{\tfrac{1}{n}}^{0}}^{1}\right) + \frac{27}{6}\cdot(1)\left(\cancel{1+\cancel{\tfrac{1}{n}}^{0}}^{1}\right)\cdot\left(\cancel{2+\cancel{\tfrac{1}{n}}^{0}}^{2}\right)$$

$\infty \quad \infty \quad \infty$

$$= 12 + 18\cdot1\cdot1 + \frac{\cancel{27}^{9}}{\cancel{6}_{\cancel{3}}}\cdot1\cdot1\cdot\cancel{2}$$

$$= 12 + 18 + 9 = 39$$ Match!