Worksheet 10 Spring 23 Answer Key

1. $y = \ln\left(\frac{1}{x}\right) + \frac{1}{\ln x} = \ln\left(\frac{1}{x}\right) + (\ln x)^{-1}$ prep

$$y' = \frac{1}{\frac{1}{\cancel{x}}}\left(-\frac{1}{x^{\cancel{2}}}\right) - (\ln x)^{-2}\,\frac{1}{x} = -\frac{1}{x} - \frac{1}{x(\ln x)^2}$$

OR, First piece option 2

$\ln\left(\frac{1}{x}\right) = \ln(x^{-1}) = -\ln x$ ↳ Derivative $-\frac{1}{x}$ Matches

2. $f(x) = (\ln x)^3 + \ln(x^3)$

$$f'(x) = 3(\ln x)^2\,\frac{1}{x} + \frac{1}{x^{\cancel{3}}}\cdot(3x^{\cancel{2}}) = \frac{3(\ln x)^2}{x} + \frac{3}{x} = \frac{3(\ln x)^2 + 3}{x}$$

3. $f(x) = \frac{1}{\sqrt{\ln x}} + \frac{1}{\ln\sqrt{x}} + \frac{1}{e^{\sqrt{x}}} + \frac{1}{e^x + \ln x}$

prep $= (\ln x)^{-\frac{1}{2}} + (\ln\sqrt{x})^{-1} + e^{-\sqrt{x}} + (e^x + \ln x)^{-1}$

$$f'(x) = \boxed{-\frac{1}{2}(\ln x)^{-3/2}\cdot\frac{1}{x} - (\ln\sqrt{x})^{-2}\cdot\frac{1}{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}} + e^{-\sqrt{x}}\cdot\left(-\frac{1}{2\sqrt{x}}\right) - (e^x + \ln x)^{-2}\cdot\left(e^x + \frac{1}{x}\right)}$$

4. Let $y = (\tan x)^x$ Take Logs of both sides for Logarithmic Differentiation

$\ln y = \ln\left((\tan x)^{\boxed{x}}\right)$ Log Algebra to pull Down Power

$\ln y = x\cdot\ln(\tan x)$

$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left(x\cdot\ln(\tan x)\right)$

$\frac{1}{y}\cdot\frac{dy}{dx} = x\,\frac{1}{\tan x}\cdot\sec^2 x + \ln(\tan x)\cdot(1)$

$$\frac{dy}{dx} = \cancel{y}^{(\tan x)^x}\left(\frac{x\sec^2 x}{\tan x} + \ln(\tan x)\right) = (\tan x)^x\left(\frac{x\sec^2 x}{\tan x} + \ln(\tan x)\right)$$

5. $y = \ln\left(\frac{\sqrt{1-x}\cdot e^{\sec x}}{(\sin x)^{6/7}\cdot\ln x}\right) = \ln\left(\sqrt{1-x}\;e^{\sec x}\right) - \ln\left((\sin x)^{6/7}\cdot\ln x\right)$

→ using Log Algebra.
→ Avoids Giant Quotient Rule

$= \ln\left((1-x)^{1/2}\right) + \cancel{\ln}\,\cancel{e}^{\sec x}\big) - \left(\ln(\sin x)^{6/7} + \ln(\ln x)\right)$

$= \frac{1}{2}\ln(1-x) + \sec x - \frac{6}{7}\ln(\sin x) - \ln(\ln x)$

$$y' = \boxed{\frac{1}{2}\cdot\left(\frac{1}{1-x}\right)(-1) + \sec x\tan x - \frac{6}{7}\cdot\frac{1}{\sin x}\cdot\cos x - \frac{1}{\ln x}\cdot\frac{1}{x}}$$

6. $y = \dfrac{\ln x}{x}$ $\qquad y' = \dfrac{\cancel{x}\left(\frac{1}{\cancel{x}}\right) - (\ln x)(1)}{x^2} = \dfrac{1-\ln x}{x^2} \overset{\text{set}}{=} 0 \Rightarrow 1 - \ln x = 0$

x^2 +

$\ln x = 1$

$x = e$ critical #

Note: y' undefined at $x=0$ but it's not a critical number because $x=0$ NOT in Domain of y

Sign Testing into Derivative

	0	1	e	3
y'		+	Max	−
y		↗		↘

Finally, y has Absolute Max Value of $f(e) = \dfrac{\cancel{\ln e}^{\,1}}{e} = \dfrac{1}{e}$ which occurs at $x = e$

7. Prove $\boxed{\dfrac{d}{dx}\ln x = \dfrac{1}{x}}$ $\qquad$ Use "L.I.D.S." Memory Aid

Let $\quad y = \ln x$

Invert $\quad e^y = \cancel{e}^{\ln x} \hookrightarrow e^y = x$

Differentiate $\quad \dfrac{d}{dx}(e^y) = \dfrac{d}{dx}(x)$

$e^y \cdot \dfrac{dy}{dx} = 1$

Solve $\quad \dfrac{dy}{dx} = \dfrac{1}{\cancel{e^y} \to x} = \dfrac{1}{x}$ ✓

8. $\displaystyle\int \frac{e^{4x}}{(1+e^{4x})^2}\,dx = \frac{1}{4}\int \frac{1}{u^2}\,du = \frac{1}{4}\int u^{-2}\,du = \frac{1}{4}\left(\frac{u^{-1}}{-1}\right) + C = \frac{-1}{4(1+e^{4x})} + C$

$u = 1 + e^{4x}$

$du = 4e^{4x}\,dx$

$\frac{1}{4}\,du = e^{4x}\,dx$

9. $\displaystyle\int \frac{(1+e^{4x})^2}{e^{4x}}\,dx = \int \frac{1+2e^{4x}+e^{8x}}{e^{4x}}\,dx = \int \frac{1}{e^{4x}} + \frac{2\cancel{e^{4x}}}{\cancel{e^{4x}}} + \frac{e^{8x}}{e^{4x}}\,dx$

split

u-sub No Match ↪ Algebra FOIL

$\cancel{u = 1+e^{4x}}$

$\cancel{du = 4e^{4}dx}$

$$= \int e^{-4x} + 2 + e^{4x}\,dx$$

$$= \frac{e^{-4x}}{-4} + 2x + \frac{e^{4x}}{4} + C$$

Now "K-rule"

$$\int e^{kx}\,dx = \frac{e^{kx}}{k} + C$$

constant

10. $\displaystyle\int_{e^3}^{e^8} \frac{8}{x\sqrt{1+\ln x}}\,dx = 8\int_4^9 \frac{1}{\sqrt{u}}\,du = 8\,\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\Bigg|_4^9 = 16\sqrt{u}\,\Bigg|_4^9$

$u^{-1/2}$

$= 16(\sqrt{9} - \sqrt{4}) = 16(3-2) = 16$

$\boxed{\begin{array}{l} u = 1+\ln x \\ du = \frac{1}{x}\,dx \end{array}}$ $\boxed{\begin{array}{l} x=e^3 \Rightarrow u = 1+\ln(e^3) = 1+3 = 4 \\ x=e^8 \Rightarrow u = 1+\ln(e^8) = 1+8 = 9 \end{array}}$

11. $\displaystyle\int_0^{\frac{\pi}{6}} \tan x\,dx = \int_0^{\frac{\pi}{6}} \frac{\sin x}{\cos x}\,dx = -\int_1^{\frac{\sqrt{3}}{2}} \frac{1}{u}\,du = -\ln|u|\,\Bigg|_1^{\frac{\sqrt{3}}{2}} = -\ln\left|\frac{\sqrt{3}}{2}\right| + \cancel{\ln|1|} = -\ln\left(\frac{\sqrt{3}}{2}\right)$

$\boxed{\begin{array}{l} u = \cos x \\ du = -\sin x\,dx \\ -du = \sin x\,dx \end{array}}$ $\boxed{\begin{array}{l} x=0 \Rightarrow u = \cos 0 = 1 \\ x=\frac{\pi}{6} \Rightarrow u = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \end{array}}$

12. $\displaystyle\int_0^{\ln 2} \frac{e^{3x}}{8+e^{3x}}\,dx = \frac{1}{3}\int_9^{16} \frac{1}{u}\,du = \frac{1}{3}\ln|u|\,\Bigg|_9^{16} = \frac{1}{3}\left(\ln|16| - \ln|9|\right) = \frac{1}{3}\ln\left(\frac{16}{9}\right)$

$\boxed{\begin{array}{l} u = 8+e^{3x} \\ du = 3e^{3x}dx \\ \frac{1}{3}du = e^{3x}dx \end{array}}$ $\boxed{\begin{array}{l} x=0 \Rightarrow u = 8+\cancel{e^{3\cdot 0}} = 9 \\ x=\ln 2 \Rightarrow u = 8+e^{3\cdot\ln 2} \\ \quad = 8+\cancel{e}^{\ln(2^3)} \\ \quad = 8+8 = 16 \end{array}}$

13. $\displaystyle\int \frac{x^6}{2-x^7}\,dx = -\frac{1}{7}\int \frac{1}{u}\,du = -\frac{1}{7}\ln|u| + C = -\frac{1}{7}\ln|2-x^7| + C$

$u = 2-x^7$

$du = -7x^6dx$

$-\frac{1}{7}du = x^6dx$

14. $\displaystyle\int \frac{2-x^6}{x^7}\,dx \overset{\text{split}}{=} \int \frac{2}{x^7} - \frac{x^6}{x^7}\,dx = \int 2x^{-7} - \frac{1}{x}\,dx = \frac{2x^{-6}}{-6} - \ln|x| + C$

$$= \boxed{-\frac{1}{3x^6} - \ln|x| + C}$$

15. $y = \ln(1+\cos x) - e\cdot\cos(\ln(1+x)) + e^{1+\ln(1+x)} + (\sin x)\cdot e^{\cos x}$

y-value: $y(0) = \ln(1+\cos 0) - e\cos(\ln(1)) + e^{1+\ln 1} + \sin 0 \cdot e^{\cos 0}$

$= \ln 2 - \cancel{e} + \cancel{e} + 0 = \ln 2$

$y' = \dfrac{1}{1+\cos x}(-\sin x) + e\sin(\ln(1+x))\cdot\left(\dfrac{1}{1+x}\right) + e^{1+\ln(1+x)}\cdot\dfrac{1}{1+x} + \sin x\cdot e^{\cos x}(-\sin x) + e^{\cos x}\cdot\cos x$

$y'(0) = -\dfrac{\sin 0}{1+\cos 0} + e\sin(\ln(1+0))\cdot\dfrac{1}{1+0} + e^{1+\ln(1+0)}\cdot\dfrac{1}{1+0} - \sin^2 0\cdot e^{\cos 0} + e^{\cos 0}\cdot\cos 0$

$= 0 + 0 + e - 0 + e = 2e$

Point - Slope Form

$y - y(0) = y'(0)(x-0)$

$y - \ln 2 = 2e(x-0) \hookrightarrow \boxed{y = 2ex + \ln 2}$