Exam 3 Spring 2022

1. $\frac{d}{dx}\left(\frac{\ln x\sqrt{1+e^x}}{(4-x^6)^3\, e^{-\cos x}}\right)$

use Log Algebra Rules

1st Simplify

$$= \frac{d}{dx}\ \ln\left(\ln x\cdot\sqrt{1+e^x}\right) - \ln\left((4-x^6)^3\, e^{-\cos x}\right)$$

$$= \frac{d}{dx}\ \ln(\ln x) + \ln\left((1+e^x)^{\frac{1}{2}}\right) - \left(\ln\left((4-x^6)^3\right) + \cancel{\ln\left(e^{-\cos x}\right)}\right)$$

$-\cos x$

$$= \frac{d}{dx}\ \ln(\ln x) + \frac{1}{2}\ln(1+e^x) - 3\ln(4-x^6) + \cos x$$

2nd Differentiation ↳ Chain Rule

$$= \frac{1}{\ln x}\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{1}{1+e^x}\cdot e^x - 3\,\frac{1}{4-x^6}\cdot(-6x^5) - \sin x$$

$$= \boxed{\frac{1}{x\ln x} + \frac{e^x}{2(1+e^x)} + \frac{18x^5}{4-x^6} - \sin x}$$

2. Logarithmic Differentiation

$y = x^x$ Take Log of both sides

$\ln y = \ln(x^x)$

$\ln y = x\cdot\ln x$

$\frac{d}{dx}(\ln y) = \frac{d}{dx}(x\cdot\ln x)$

$\frac{1}{y}\cdot\frac{dy}{dx} = \cancel{x\cdot\frac{1}{x}} + \ln x\cdot(1)$

$$\frac{dy}{dx} = \cancel{y}^{\,x^x}(1+\ln x) = \boxed{x^x(1+\ln x)}$$

3(a) $f(x)=\sqrt{\ln x} - \ln\sqrt{x}$

OR// 2nd piece

$\ln\sqrt{x} = \ln(x^{1/2}) = \frac{1}{2}\ln x$

$\hookrightarrow \frac{1}{2}\cdot\frac{1}{x}$ Match

$$f'(x) = \frac{1}{2\sqrt{\ln x}}\cdot\frac{1}{x} - \frac{1}{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}$$

$2x$

$$f'(e^4) = \frac{1}{2\sqrt{\underbrace{\cancel{\ln(e^4)}}_{4}}\cdot e^4} - \frac{1}{2e^4} = \frac{1}{4e^4} - \frac{2}{4e^4} = \frac{-1}{4e^4}$$

$\sqrt{4}=2$

Match!

3(b) $f(x)=e^{2x}+\frac{1}{e^{2x}} = e^{2x}+e^{-2x}$

$$f'(x)=e^{2x}\cdot 2 - 2e^{-2x} = 2e^{2x} - \frac{2}{e^{2x}}$$

$$f'(\ln 3) = 2e^{2\ln 3} - \frac{2}{e^{2\ln 3}}$$

$$= 2\overset{9}{\cancel{e^{\ln(3^2)}}} - \frac{2}{\cancel{e^{\ln(3^2)}}\ 9}$$

$$= 18 - \frac{2}{9} = \frac{162}{9} - \frac{2}{9} = \frac{160}{9}$$

Match!

4. $f(x)=\sin(\ln(1+x)) - \ln(1+\sin(5x)) - e^{\cos x} - \sin(e^{3x}-1)$

$$f'(x) = \cos(\ln(x+1))\cdot\frac{1}{x+1} - \frac{1}{1+\sin(5x)}\cdot\cos(5x)\cdot 5 - e^{\cos x}\cdot(-\sin x) - \cos(e^{3x}-1)\cdot e^{3x}\cdot 3$$

$$f'(0) = \overset{1}{\cos(\overset{0}{\ln 1})}\cdot\frac{1}{1} - \frac{1}{1+\underset{0}{\sin 0}}\cdot\overset{1}{\cos 0}\cdot 5 \overset{--}{+} e^{\overset{1}{\cos 0}}\cdot\overset{0}{\sin 0} - \cos(\overset{1}{e^0}-1)\,\overset{1}{e^0}\cdot 3$$

$5 \quad e \quad 0 \quad 0 \quad 3$

$$= 1 - 5 + 0 - 3 = -7$$

Match!

Constant $\qquad$ Constant

$$5. \quad f(x) = e^{6x} + \frac{6}{e^{6x}} + \cancel{e^{\ln 6}} - \frac{6}{x} + \frac{1}{e^{6x}} - \cancel{\ln(e^6)} + 6e^{6x} - \frac{e}{x^6} + \frac{e^x}{e^{6x}} + (e^{6x})(e^x)$$

Prep

$$= e^{6x} + 6e^{-6x} + 6 - 6x^{-1} + e^{-6x} - 6 + 6e^{6x} - e \cdot x^{-6} + e^{-5x} + e^{7x}$$

$$f'(x) = \boxed{6e^{6x} - 36e^{-6x} + 0 + 6x^{-2} - 6e^{-6x} + 0 + 36e^{6x} + 6e\,x^{-7} - 5e^{-5x} + 7e^{7x}}$$

$$6(a) \quad \int e^{2x}(2+e^{2x})^7\,dx = \frac{1}{2}\int u^7\,du = \frac{1}{2}\cdot\frac{u^8}{8} + C = \frac{u^8}{16} + C$$

$u = 2 + e^{2x}$

$du = 2e^{2x}\,dx$

$\frac{1}{2}\,du = e^{2x}\,dx$

$$= \boxed{\frac{(2+e^{2x})^8}{16} + C}$$

$$6(b) \quad \int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int e^u\,du = 2e^u + C = \boxed{2e^{\sqrt{x}} + C}$$

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$2\,du = \frac{1}{\sqrt{x}}\,dx$

split

$$6(c) \quad \int \frac{(1+e^{2x})^2}{e^{6x}}\,dx = \int \frac{1 + 2e^{2x} + e^{4x}}{e^{6x}}\,dx = \int e^{-6x} + 2e^{-4x} + e^{-2x}\,dx$$

FOIL Algebra
+
Split-Split

$$= \frac{e^{-6x}}{-6} + \frac{\cancel{2}e^{-4x}}{-\cancel{4}_{2}} + \frac{e^{-2x}}{-2} + C$$

using k-rule

$$\int e^{kx}\,dx = \frac{e^{kx}}{k} + C$$

OR

$$= \boxed{-\frac{1}{6e^{6x}} - \frac{1}{2e^{4x}} - \frac{1}{2e^{2x}} + C}$$

7(a) $\displaystyle\int_{\frac{\pi}{2}}^{\pi} \frac{\sin x}{e+\cos x}\,dx = -\int_{e}^{e-1} \frac{1}{u}\,du = -\ln|u|\Big|_{e}^{e-1}$

$$= -\ln|e-1| + \cancel{\ln|e|}^{\,1}$$

$$= \boxed{1-\ln|e-1|}$$

Match!

$$\boxed{\begin{array}{l} u = e+\cos x \\ du = -\sin x\,dx \\ -du = \sin x\,dx \end{array}}$$

$$\boxed{\begin{array}{l} x=\frac{\pi}{2} \Rightarrow u = e+\cancel{\cos\frac{\pi}{2}}^{\,0} = e \\ x=\pi \Rightarrow u = e+\cancel{\cos\pi}^{\,-1} = e-1 \end{array}}$$

7(b) $\displaystyle\int_{0}^{\ln 3} \frac{e^{2x}}{1+e^{2x}}\,dx = \frac{1}{2}\int_{2}^{10} \frac{1}{u}\,du = \frac{1}{2}\ln|u|\Big|_{2}^{10} = \frac{1}{2}\left(\ln(10)-\ln 2\right)$

$$= \frac{1}{2}\ln\left(\frac{10}{2}\right)$$

$$= \frac{1}{2}\ln 5$$

Match!

$$\boxed{\begin{array}{l} u = 1+e^{2x} \\ du = 2e^{2x}\,dx \\ \frac{1}{2}du = e^{2x}\,dx \end{array}}$$

$$\boxed{\begin{array}{l} x=0 \Rightarrow u = 1+e^{0} = 2 \\ x=\ln 3 \Rightarrow u = 1+e^{2\ln 3} \\ \qquad = 1+\cancel{e^{\ln(3^2)}}^{\,9} = 10 \end{array}}$$

7(c) $\displaystyle\int_{e^3}^{e^8} \frac{1}{x\sqrt{1+\ln x}}\,dx = \int_{4}^{9} \frac{1}{\sqrt{u}}\,du = \int_{4}^{9} u^{-1/2}\,du = \frac{u^{1/2}}{1/2}\Big|_{4}^{9} = 2\sqrt{u}\Big|_{4}^{9}$

$$= 2\left(\cancel{\sqrt{9}}^{\,3} - \cancel{\sqrt{4}}^{\,2}\right) = 2(3-2) = 2$$

Match!

$$\boxed{\begin{array}{l} u = 1+\ln x \\ du = \frac{1}{x}\,dx \end{array}}$$

$$\boxed{\begin{array}{l} x=e^3 \Rightarrow u = 1+\cancel{\ln}\,e^3 = 4 \\ x=e^8 \Rightarrow u = 1+\ln e^8 = 9 \end{array}}$$