Exam 3 Spring 2018 Answer Key

1(a) $$y = \ln\left(\frac{\ln x\,\sqrt{1+e^x}}{(4-x^6)^3\, e^{-\cos x}}\right)$$ simplify 1st using Log Properties

$$= \ln\left(\ln x\cdot\sqrt{1+e^x}\right) - \ln\left((4-x^6)^3\cdot e^{-\cos x}\right)$$ $\ln\left(\frac{A}{B}\right) = \ln A - \ln B$

$$= \ln(\ln x) + \ln\left[(1+e^x)^{1/2}\right] - \left[\ln\left((4-x^6)^3\right) + \cancel{\ln e}^{-\cos x}\right]$$ $\ln(A\cdot B) = \ln A + \ln B$

$$= \ln(\ln x) + \frac{1}{2}\underbrace{\ln(1+e^x)}_{\text{does not simplify}} - 3\ln(4-x^6) + \cos x$$ $\ln\left(A^B\right) = B\cdot\ln A$

$$y' = \boxed{\frac{1}{\ln x}\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{1}{1+e^x}\cdot e^x - 3\cdot\frac{1}{4-x^6}\cdot(-6x^5) - \sin x}$$

1(b) $$\frac{d}{dx}(\cos x)^{\sin x}$$

Let $y = (\cos x)^{\sin x}$

$$\ln y = \ln\left((\cos x)^{\sin x}\right)$$

$$\ln y = \sin x\cdot\ln(\cos x)$$

Differentiate

$$\frac{d}{dx}\ln y = \frac{d}{dx}\left(\sin x\cdot\ln(\cos x)\right)$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \sin x\cdot\frac{1}{\cos x}\cdot(-\sin x) + \ln(\cos x)\cdot\cos x$$

Solve $$\frac{dy}{dx} = \cancel{y}^{\,(\cos x)^{\sin x}}\left(-\frac{\sin^2 x}{\cos x} + \cos x\cdot\ln(\cos x)\right)$$

$$= \boxed{(\cos x)^{x}\cdot\left(-\frac{\sin^2 x}{\cos x} + \cos x\cdot\ln(\cos x)\right)}$$

Resubstitute for y

recall: $\sin^2 x = (\sin x)^2 = \sin x\cdot\sin x$

1(c) $y = \ln(\ln(\cancel{\ln e}^{x})) + \frac{e}{\ln x} + \frac{\ln x}{e} + \ln x\cdot e^x + \underbrace{\ln(xe^x)}_{\text{split}}$ simplify 1st?

$$= \ln(\ln x) + \overset{\text{prep}}{e(\ln x)^{-1}} + \frac{1}{e}\cdot\ln x + \ln x\cdot e^x + \ln x + \underbrace{\cancel{\ln(e}^{x})}_{x}$$

$$\boxed{y' = \frac{1}{\ln x}\cdot\frac{1}{x} - e(\ln x)^{-2}\cdot\frac{1}{x} + \frac{1}{e}\cdot\frac{1}{x} + \left(\ln x\cdot e^x + e^x\cdot\frac{1}{x}\right) + \frac{1}{x} + 1}$$

1(d) $y = \underbrace{\cancel{e}^{\cancel{\ln}(\ln x)}}_{\ln x} + \underbrace{\overbrace{\cancel{\ln}(\cancel{e}^{5})}^{\text{constant}}}_{5} + \cancel{e}^{\cancel{\ln} x} + \underbrace{\overbrace{5^{\cancel{\ln e}}}^{\text{constant}}}_{5}$

$$= \ln x + 5 + x + 5$$

$$y' = \frac{1}{x} + 0 + 1 + 0 = \boxed{\frac{1}{x} + 1}$$

OR Note: If you don't simplify 1st then y' looks as follows

$$y' = \underbrace{\cancel{e}^{\cancel{\ln}(\ln x)}}\cdot\frac{1}{\ln x}\cdot\frac{1}{x} + \frac{1}{\cancel{e^5}}\cdot\cancel{e^5}\cdot 0 + \cancel{e}^{\cancel{\ln} x}\cdot\frac{1}{x} + 0$$

$$= \cancel{\ln x}\cdot\frac{1}{\cancel{\ln x}}\cdot\frac{1}{x} + 0 + \cancel{x}\cdot\frac{1}{\cancel{x}}$$

$$= \frac{1}{x} + 1$$ Match!

2(a) $f(x) = \frac{x+2}{e^x}$

factor off common factor

$$f'(x) = \frac{e^x(1) - (x+2)e^x}{(e^x)^2} = \frac{\cancel{e^x}(1 - x - 2)}{(e^x)^{\cancel{2}}} = \frac{-x-1}{e^x} \overset{\text{set}}{=} 0 \Rightarrow -x-1 = 0$$

$x = -1$ critical #

e^x: + denominator

Sign Testing into $f'(x)$

	-2	-1	0
f'	$+$		$-$
f	$\nearrow$	Local MAX	$\searrow$

Local Min: None

Local Max Value of $f(-1) = \frac{1}{e^{-1}} = e$

occurs at $x = -1$.

2(b) $y = \left[\ln(x+4)\right]^2$

$$y' = 2\,\frac{\ln(x+4)}{x+4} \overset{\text{set}}{=} 0$$

$$2\ln(x+4) = 0$$

$$\ln(x+4) = 0$$

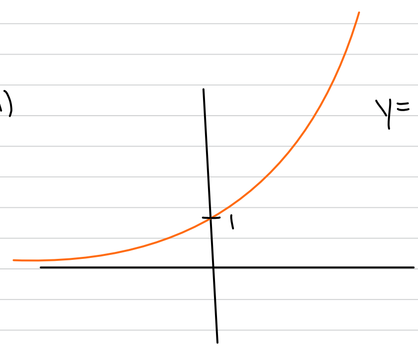
$$e^{\cancel{\ln}(x+4)} = \cancel{e^{0}}\ 1$$

$$x+4 = 1 \hookrightarrow x = -3$$

y-value @ $x=-3$ given by $y(-3) = \left[\ln(\cancel{-3+4}^{\,1})\right]^2 = (\cancel{\ln 1}^{\,0})^2 = 0$

Point: $(-3, y(-3)) = \boxed{(-3,0)}$ Note: problem asked for Point

3(a)



$y = e^x$

Domain $= \mathbb{R} = (-\infty, \infty)$

Range $= (0,\infty) = \{y : y > 0\}$

3(b)

$y = \ln x$

Domain $= (0,\infty) = \{x : x > 0\}$

Range $= \mathbb{R} = (-\infty, \infty)$

4(a)

K-rule Right Awa

$$\int \frac{1}{x^3 e^{1/x^2}}\,dx = -\frac{1}{2}\int \frac{1}{e^u}\,du = -\frac{1}{2}\int e^{-u}\,du = \cancel{-}\frac{1}{2}\cdot\frac{e^{-u}}{\cancel{-1}} + C$$

$$= \boxed{\frac{1}{2e^{1/x^2}} + C}$$

$$\boxed{\begin{aligned} u &= \frac{1}{x^2} \\ du &= -\frac{2}{x^3}\,dx \\ -\frac{1}{2}\,du &= \frac{1}{x^3}\,dx \end{aligned}}$$

4(b) $\displaystyle\int_1^{\sqrt{6}} \frac{x}{7-x^2}\,dx = -\frac{1}{2}\int_6^1 \frac{1}{u}\,du = -\frac{1}{2}\ln|u|\Big|_6^1 = -\frac{1}{2}\left(\cancel{\ln 1}^{0} - \ln 6\right)$

$= \cancel{-}\frac{1}{2}\left(\cancel{-}\ln 6\right)$

$= \boxed{\frac{1}{2}\ln 6} \overset{\text{OR}}{=} \frac{\ln 6}{2}$

$\overset{\text{OR}}{=} \ln\sqrt{6}$

$u = 7-x^2$
$du = -2x\,dx$
$-\frac{1}{2}\,du = x\,dx$

$x=1 \Rightarrow u = 7-1 = 6$
$x=\sqrt{6} \Rightarrow u = 7-(\sqrt{6})^2 = 7-6 = 1$

4(c) $\displaystyle\int_{e^3}^{e^8} \frac{4}{x\sqrt{1+\ln x}}\,dx = \int_4^9 \frac{4}{\sqrt{u}}\,du = 4\frac{u^{1/2}}{1/2}\Big|_4^9 = 8\sqrt{u}\Big|_4^9$

$u^{-1/2}$

$= 8\left(\cancel{\sqrt{9}}^{3} - \cancel{\sqrt{4}}^{2}\right)$

$= 8(3-2) = \boxed{8}$

$u = 1+\ln x$
$du = \frac{1}{x}\,dx$

$x = e^3 \Rightarrow u = 1+\cancel{\ln}\, e^3 = 4$
$x = e^8 \Rightarrow u = 1+\cancel{\ln}\, e^8 = 9$

4(d) $\displaystyle\int_{-3}^{-1} \frac{1-x}{x^2}\,dx = \int_{-3}^{-1} \frac{1}{x^2} - \frac{x}{x^2}\,dx = \int_{-3}^{-1} x^{-2} - \frac{1}{x}\,dx = \frac{x^{-1}}{-1} - \ln|x|\Big|_{-3}^{-1}$

split

$= \frac{-1}{x} - \ln|x|\Big|_{-3}^{-1}$

Must have $|\cdot|$ Absolute Values

$= \cancel{-}\frac{1}{\cancel{-}1} - \cancel{\ln|-1|}^{0} - \left(\cancel{-}\frac{1}{\cancel{-}3} - \ln|-3|\right)$

$1 \qquad 3$

$= 1 - \frac{1}{3} + \ln 3 = \boxed{\frac{2}{3} + \ln 3}$

4(e) $\displaystyle\int_{\pi/18}^{\pi/9} \tan(3x)\,dx = \int_{\pi/18}^{\pi/9} \frac{\sin(3x)}{\cos(3x)}\,dx = -\frac{1}{3}\int_{\frac{\sqrt{3}}{2}}^{1/2} \frac{1}{u}\,du = -\frac{1}{3}\ln|u|\Big|_{\frac{\sqrt{3}}{2}}^{\frac{1}{2}}$

$u = \cos(3x)$
$du = -\sin(3x)\cdot 3\,dx$
$-\frac{1}{3}\,du = \sin(3x)\,dx$

$x = \frac{\pi}{18} \Rightarrow u = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$
$x = \frac{\pi}{9} \Rightarrow u = \cos\frac{\pi}{3} = \frac{1}{2}$

$= -\frac{1}{3}\left(\ln\left(\frac{1}{2}\right) - \ln\left(\frac{\sqrt{3}}{2}\right)\right)$

$= -\frac{1}{3}\left(\cancel{\ln 1}^{0} - \cancel{\ln 2} - \ln\sqrt{3} \overset{--}{+} \cancel{\ln 2}\right)$

$= \cancel{-}\frac{1}{3}\left(\cancel{-}\ln\sqrt{3}\right)$

$= \frac{1}{3}\ln\left(3^{1/2}\right) = \frac{1}{6}\cdot\ln 3 = \boxed{\frac{\ln 3}{6}}$

Match!

4(f) $\displaystyle\int_0^{\ln 2}\left(e^x+\frac{1}{e^{2x}}\right)^2 dx = \int_0^{\ln 2}\left(e^x+\frac{1}{e^{2x}}\right)\left(e^x+\frac{1}{e^{2x}}\right)dx$

FOIL

$$= \int_0^{\ln 2} e^{2x}+e^{-x}+e^{-x}+\frac{1}{e^{4x}}\,dx$$

K-rule

$$\int e^{kx}dx = \frac{e^{kx}}{k}+C$$

$$= \int_0^{\ln 2} e^{2x}+2e^{-x}+e^{-4x}\,dx$$

$$= \frac{e^{2x}}{2}+\frac{2e^{-x}}{-1}+\frac{e^{-4x}}{-4}\Bigg|_0^{\ln 2}$$

$$= \frac{e^{2\ln 2}}{2}-2e^{-\ln 2}-\frac{e^{-4\ln 2}}{4}-\left(\frac{e^{0}}{2}-2e^{0}-\frac{1}{4}e^{0}\right)$$

$$= \frac{1}{2}\cancel{e}^{\cancel{\ln}(2^2)}-2\cancel{e}^{\cancel{\ln}(2^{-1})}-\frac{1}{4}\cancel{e}^{\cancel{\ln}(2^{-4})} \quad -\frac{1}{2}+2+\frac{1}{4}$$

$$= \frac{1}{\cancel{2}}\cdot\cancel{4}^{2}-\cancel{2}\cdot\frac{1}{\cancel{2}}-\frac{1}{4}\cdot\frac{1}{16}-\frac{1}{2}+2+\frac{1}{4}$$

$$= 2-1-\frac{1}{64}-\frac{1}{2}+2+\frac{1}{4}$$

Match!

$$= 3-\frac{1}{64}-\frac{1}{2}+\frac{1}{4} = \frac{192}{64}-\frac{1}{64}-\frac{32}{64}+\frac{16}{64} = \boxed{\frac{175}{64}}$$

$$\begin{array}{r} 64 \\ \times\ 3 \\ \hline 192 \end{array}$$

191   −16

5. $f(x) = \int f'(x)\,dx = \int \frac{1}{e^{2x}(1-2e^{-2x})^2}\,dx$

$\boxed{\begin{aligned} u &= 1-2e^{-2x} \\ du &= 4e^{-2x}dx \\ \tfrac{1}{4}du &= \tfrac{1}{e^{2x}}dx \end{aligned}}$

$$= \frac{1}{4}\int \frac{1}{u^2}\,du \quad u^{-2}$$

$$= \frac{1}{4}\left(\frac{u^{-1}}{-1}\right) + C$$

$$= \frac{-1}{4u} + C$$

$$= -\frac{1}{4(1-2e^{-2x})} + C$$

Test Initial Value to Solve for +C

$$f(0) = \frac{-1}{4(1-2\cancel{e^0}_{1})} + C \overset{\text{set}}{=} -1$$

$$\cancel{-}\frac{1}{4(\cancel{-}1)} + C = -1 \hookrightarrow \frac{1}{4} + C = -1 \hookrightarrow C = -\frac{5}{4}$$

Finally, $\boxed{f(x) = -\frac{1}{4(1-2e^{-2x})} - \frac{5}{4}}$

6. Prove $\frac{d}{dx}\ln x = \frac{1}{x}$

Let $y = \ln x$

Invert $e^y = \cancel{e}^{\cancel{\ln} x}$

$e^y = x$

Differentiate both Sides

$$\frac{d}{dx}(e^y) = \frac{d}{dx}(x)$$

$$e^y \cdot \frac{dy}{dx} = 1$$

Solve $\frac{dy}{dx} = \frac{1}{\cancel{e^y}\to x} = \frac{1}{x}$ ✓