Exam 2 Spring 2022

1(a) $\int_{-1}^{2} 2-2x-x^2\,dx = 2x - x^2 - \frac{x^3}{3}\Big|_{-1}^{2} = 4 - 2^2 - \frac{2^3}{3} - \left(-2 - (-1)^2 - \frac{(-1)^3}{3}\right)$

$= \cancel{4} - \cancel{4} - \frac{8}{3} + 2 + 1 - \frac{1}{3}$

$= 3 - \cancel{\frac{9}{3}}^{\,3} = 3 - 3 = 0$ Match!

1(b) $\int_{-1}^{2} 2-2x-x^2\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$

$a=-1 \quad b=2$

$f(x) = 2-2x-x^2$

$\Delta x = \frac{b-a}{n} = \frac{2-(-1)}{n} = \frac{3}{n}$

$x_i = a + i\Delta x = -1 + \frac{3i}{n}$

$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(-1+\frac{3i}{n}\right)\cdot\frac{3}{n}$ $\quad -\left(1 - \frac{6i}{n} + \frac{9i^2}{n^2}\right)$

$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} 2 - 2\left(-1+\frac{3i}{n}\right) - \left(-1+\frac{3i}{n}\right)^2$

$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} 2 + 2 - \cancel{\frac{6i}{n}} - 1 + \cancel{\frac{6i}{n}} - \frac{9i^2}{n^2}$

$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} 3 - \frac{9i^2}{n^2}$

$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} 3 - \frac{3}{n} \sum_{i=1}^{n} \frac{9i^2}{n^2}$

$= \lim_{n\to\infty} \frac{9}{n}\cdot \cancel{\sum_{i=1}^{n} 1}^{\,n} - \frac{27}{n^3} \cancel{\sum_{i=1}^{n} i^2}$

$= \lim_{n\to\infty} \frac{9}{\cancel{n}}\cdot\cancel{n} - \frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$

$= \lim_{n\to\infty} 9 - \frac{27}{6}\left(\cancel{\frac{n}{n}}\right)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)$

$= \lim_{n\to\infty} 9 - \frac{27}{6}\cdot(1)\cdot\left(\cancel{1+\frac{1}{n}}^{\,1}\right)\left(\cancel{2+\frac{1}{n}}^{\,2}\right)$

$= 9 - \frac{\cancel{27}^{\,9}}{\cancel{6}_{\,3}}\cdot\cancel{2} = 9 - 9 = 0$ Match

2(a) $\displaystyle\int \frac{x^6}{(8-x^7)^5}\,dx = -\frac{1}{7}\int \frac{1}{u^5}\,du = -\frac{1}{7}\int u^{-5}\,du = \cancel{-}\frac{1}{7}\left(\frac{u^{-4}}{\cancel{-}4}\right) + C$

$$\boxed{\begin{aligned} u &= 8-x^7 \\ du &= -7x^6\,dx \\ -\tfrac{1}{7}\,du &= x^6\,dx \end{aligned}}$$

$$= \frac{1}{28u^4} + C$$

$$= \boxed{\frac{1}{28(8-x^7)^4} + C}$$

2(b) $\displaystyle\int 4\sin x\cdot\cos^3 x\,dx = 4\int \sin x\cdot(\cos x)^3\,dx$

$$\boxed{\begin{aligned} u &= \cos x \\ du &= -\sin x\,dx \\ -du &= \sin x\,dx \end{aligned}}$$

$$= -4\int u^3\,du$$

$$= -\cancel{4}\left(\frac{u^4}{\cancel{4}}\right) + C$$

$$= \boxed{-\cos^4 x + C}$$

2(c) $\displaystyle\int \sec^2(1-7x)\,dx = -\frac{1}{7}\int \sec^2 u\,du$

$$\boxed{\begin{aligned} u &= 1-7x \\ du &= -7\,dx \\ -\tfrac{1}{7}\,du &= dx \end{aligned}}$$

$$= -\frac{1}{7}\tan u + C$$

$$= \boxed{-\frac{1}{7}\tan(1-7x) + C}$$

2(d) $\displaystyle\int x(x+7)^6\,dx = \int (u-7)\cdot u^6\,du = \int u^7 - 7u^6\,du$

$$\boxed{\begin{aligned} &u = x+7 \Rightarrow x = u-7 \\ &du = dx \end{aligned}}$$

$$= \frac{u^8}{8} - \frac{\cancel{7}u^7}{\cancel{7}} + C$$

$$= \boxed{\frac{(x+7)^8}{8} - (x+7)^7 + C}$$

3(a) $\int_1^4 \frac{(\sqrt{x}-1)(\sqrt{x}+1)}{\sqrt{x}}\,dx = \int_1^4 \frac{x+\cancel{\sqrt{x}}-\cancel{\sqrt{x}}-1}{\sqrt{x}}\,dx = \int_1^4 \frac{x-1}{\sqrt{x}}\,dx$

Algebra: FOIL

$$= \int_1^4 \frac{x}{\sqrt{x}} - \frac{1}{\sqrt{x}}\,dx = \int_1^4 x^{1/2} - x^{-1/2}\,dx$$

$$= \frac{x^{3/2}}{3/2} - \frac{x^{1/2}}{1/2}\Bigg|_1^4 = \frac{2}{3}x^{\frac{3}{2}} - 2\sqrt{x}\Bigg|_1^4$$

$4^{3/2} = (\sqrt{4})^3 = 2^3 = 8$

$$= \frac{2}{3}\cdot \cancelto{8}{4^{\frac{3}{2}}} - 2\cancelto{2}{\sqrt{4}} - \left(\frac{2}{3}\cdot\cancelto{1}{1^{\frac{3}{2}}} - 2\cancelto{1}{\sqrt{1}}\right)$$

$$= \frac{16}{3} - 4 - \frac{2}{3} + 2 = \frac{14}{3} - 2 = \frac{14}{3} - \frac{6}{3} = \frac{8}{3}$$

Match!

3(b) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2 x}{\tan^3 x}\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2 x}{(\tan x)^3}\,dx = \int_1^{\sqrt{3}} \frac{1}{u^3}\,du = \int_1^{\sqrt{3}} u^{-3}\,du = \frac{u^{-2}}{-2}\Bigg|_1^{\sqrt{3}}$

prep

$\boxed{\begin{array}{l} u = \tan x \\ du = \sec^2 x\,dx \end{array}}$

$$= -\frac{1}{2u^2}\Bigg|_1^{\sqrt{3}} = -\frac{1}{2\cancelto{3}{(\sqrt{3})^2}} + \frac{1}{2}$$

$\boxed{\begin{array}{l} x = \frac{\pi}{4} \Rightarrow u = \tan\frac{\pi}{4} = 1 \\ x = \frac{\pi}{3} \Rightarrow u = \tan\frac{\pi}{3} = \sqrt{3} \end{array}}$

$$= -\frac{1}{6} + \frac{1}{2} = -\frac{1}{6} + \frac{3}{6} = \frac{2}{6} = \frac{1}{3}$$

Match!

3(c) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos(4x)\,dx = \frac{1}{4}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \cos u\,du = \frac{1}{4}\sin u\Bigg|_{\frac{2\pi}{3}}^{\frac{4\pi}{3}}$

$\boxed{\begin{array}{l} u = 4x \\ du = 4dx \\ \frac{1}{4}du = dx \end{array}}$

$$= \frac{1}{4}\left(\cancelto{-\frac{\sqrt{3}}{2}}{\sin\frac{4\pi}{3}} - \cancelto{\frac{\sqrt{3}}{2}}{\sin\frac{2\pi}{3}}\right)$$

$\boxed{\begin{array}{l} x = \frac{\pi}{6} \Rightarrow u = 4\left(\frac{\pi}{6}\right) = \frac{2\pi}{3} \\ x = \frac{\pi}{3} \Rightarrow u = \frac{4\pi}{3} \end{array}}$

$$= \frac{1}{4}\left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) = \frac{1}{4}\left(-\frac{\cancel{2}\sqrt{3}}{\cancel{2}}\right) = -\frac{\sqrt{3}}{4}$$

Match!

3(d) $\int_{2\pi}^{6\pi} \sin\left(\frac{x}{6}\right)dx = 6\int_{\frac{\pi}{3}}^{\pi} \sin u\,du = -6\cos u\Bigg|_{\frac{\pi}{3}}^{\pi} = -6\left(\cancelto{-1}{\cos\pi} - \cancelto{\frac{1}{2}}{\cos\left(\frac{\pi}{3}\right)}\right)$

$$= -6\left(-1-\frac{1}{2}\right) = -\cancelto{3}{6}\left(-\frac{3}{\cancel{2}}\right) = 9$$

$\boxed{\begin{array}{l} u = \frac{x}{6} \\ du = \frac{1}{6}dx \\ 6\,du = dx \end{array}}$

Match!

$\boxed{\begin{array}{l} x = 2\pi \Rightarrow u = \frac{2\pi}{6} = \frac{\pi}{3} \\ x = 6\pi \Rightarrow u = \frac{6\pi}{6} = \pi \end{array}}$

4. $\displaystyle\int_{-1}^{3} |x-2|+1\,dx = \int_{-1}^{2} \overbrace{|x-2|}^{-x+2}+1\,dx + \int_{2}^{3} \overbrace{|x-2|}^{x-2}+1\,dx$

Cases

$$|x-2| = \begin{cases} x-2 & \text{if } x-2 \geq 0 \\ -(x-2) & \text{if } x-2 < 0 \end{cases}$$

$$= \begin{cases} x-2 & \text{if } x \geq 2 \\ -x+2 & \text{if } x < 2 \end{cases}$$

$$= \int_{-1}^{2} -x+3\,dx + \int_{2}^{3} x-1\,dx$$

$$= -\frac{x^2}{2} + 3x \Big|_{-1}^{2} + \frac{x^2}{2} - x \Big|_{2}^{3}$$

$$= -\frac{4}{2} + 6 - \left(-\frac{\overset{1}{\cancel{(-1)^2}}}{2} - 3\right) + \frac{9}{2} - 3 - \xcancel{\left(\overset{2}{\cancel{\tfrac{4}{2}}} - 2\right)}^{\,2-2}$$

$$= -2 + 6 + \frac{1}{2} + \cancel{3} + \frac{9}{2} - \cancel{3}$$

$$= 4 + \frac{10}{2} = 4 + 5 = 9$$ Match!

5. $\displaystyle f(x) = \int f'(x)\,dx = \int \frac{1}{\sqrt{x}\sqrt{2+\sqrt{x}}}\,dx$

$u = 2+\sqrt{x}$

$du = \dfrac{1}{2\sqrt{x}}\,dx$

$2\,du = \dfrac{1}{\sqrt{x}}\,dx$

$$= 2\int \frac{1}{\sqrt{u}}\,du = 2\int u^{-1/2}\,du = 2\,\frac{u^{1/2}}{1/2} + C$$

$$= 4\sqrt{u} + C = 4\sqrt{2+\sqrt{x}} + C$$

$$f(4) = 4\sqrt{2+\underset{2}{\sqrt{4}}} + C = 4\overset{2}{\sqrt{4}} + C \overset{\text{set}}{=} -5$$

$8 + C = -5$

$\hookrightarrow C = -13$

Finally, $\boxed{f(x) = 4\sqrt{2+\sqrt{x}} - 13}$