Exam 1 Spring 2024

1(a) $f(x) = \overset{\text{constant}}{\sin\left(\frac{\pi}{4}\right)} + \sqrt{\sin\sqrt{x}} = \sin\left(\frac{\pi}{4}\right) + \left(\sin\sqrt{x}\right)^{1/2}$

$f'(x) = \boxed{0 + \frac{1}{2\sqrt{\sin\sqrt{x}}} \cdot \cos\sqrt{x} \cdot \frac{1}{2\sqrt{x}}}$

Do Not Need to Simplify

1(b) $f(x) = \cos(\sin x)$

$f'(x) = \boxed{-\sin(\sin x) \cdot \cos x}$

1(c) $f(x) = \cos x \cdot \sin x$

$f'(x) = \boxed{\cos x \cdot \cos x + \sin x(-\sin x)} \overset{\text{OR}}{=} \cos^2 x - \sin^2 x$

1(d) $f(x) = \tan^8\left(\frac{4}{x^5}\right) \overset{\text{prep}}{=} \left(\tan\left(\frac{4}{x^5}\right)\right)^8$

$\frac{4}{x^5} = 4x^{-5}$

$f'(x) = \boxed{8\left(\tan\left(\frac{4}{x^5}\right)\right)^7 \cdot \sec^2\left(\frac{4}{x^5}\right) \cdot \left(-20x^{-6}\right)}$

1(e) $f(x) = \frac{1}{\cos x} = (\cos x)^{-1}$

$f'(x) = \boxed{-(\cos x)^{-2} \cdot (-\sin x)} \overset{\text{OR}}{=} \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \cdot \tan x$

OR $f(x) = \frac{1}{\cos x} = \sec x \Rightarrow f'(x) = \sec x \cdot \tan x$

(1f) $f(x) = \cos\left(\frac{1}{x}\right)$

$f'(x) = \boxed{-\sin\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right)} = \sin\left(\frac{1}{x}\right) \cdot \frac{1}{x^2}$

2(a) $f(x) = \cos(7x) + \cos(6x) + \sin(3x) + \sin(4x)$

$f'(x) = -7\sin(7x) - 6\sin(6x) + 3\cos(3x) + 4\cos(4x)$

$f'\left(\frac{\pi}{6}\right) = -7\sin\left(\frac{7\pi}{6}\right) - 6\sin\left(\cancel{6} \cdot \frac{\pi}{\cancel{6}}\right) + 3\cos\left(\cancel{3}\frac{\pi}{\cancel{6}_2}\right) + 4\cos\left(\cancel{4}^2 \cdot \frac{\pi}{\cancel{6}_3}\right)$

$= \underbrace{-7\sin\left(\frac{7\pi}{6}\right)}_{\text{See 4. above}} \overset{-\frac{1}{2}}{} - 6\sin(\pi)\overset{0}{} + 3\cos\left(\frac{\pi}{2}\right)\overset{0}{} + \underbrace{4\cos\left(\frac{2\pi}{3}\right)}_{\text{See 5. above}}\overset{-\frac{1}{2}}{}$

$= \frac{7}{2} + 0 + 0 - 2 = \frac{7}{2} - 2 = \frac{7}{2} - \frac{4}{2} = \frac{3}{2}$ Match

2(b) $H(x) = \cos^2(2x) + \sin(6x) + 2\sin x$

$= (\cos(2x))^2 + \sin(6x) + 2\sin x$

$H'(x) = 2\cos(2x)\cdot(-\sin(2x))\cdot 2 + 6\cos(6x) + 2\cos x$

$H'\left(\frac{\pi}{6}\right) = -4\cos\left(\cancel{2}\cdot\frac{\pi}{\cancel{6}_3}\right)\cdot\sin\left(\cancel{2}\cdot\frac{\pi}{\cancel{6}_3}\right) + 6\cos\left(\cancel{6}\cdot\frac{\pi}{\cancel{6}}\right) + 2\cos\left(\frac{\pi}{6}\right)$

$= -4\cancel{\cos\left(\frac{\pi}{3}\right)}^{\frac{1}{2}}\cdot\cancel{\sin\left(\frac{\pi}{3}\right)}^{\frac{\sqrt{3}}{2}} + 6\cancel{\cos\pi}^{-1} + 2\cancel{\cos\left(\frac{\pi}{6}\right)}^{\frac{\sqrt{3}}{2}}$

$= -\cancel{4}\left(\frac{1}{\cancel{2}}\right)\left(\frac{\sqrt{3}}{\cancel{2}}\right) + 6(-1) + \cancel{2}\cdot\frac{\sqrt{3}}{\cancel{2}}$

$= -\cancel{\sqrt{3}} - 6 + \cancel{\sqrt{3}} = -6$ Match!

3(a) $\int \frac{5}{6}x^5 + x^{\frac{5}{6}} + \frac{1}{x^{\frac{5}{6}}} + \frac{6}{5} + \frac{5}{6x^6} - \frac{6}{x^5}\,dx$

prep

$= \int \frac{5}{6}x^5 + x^{\frac{5}{6}} + x^{\frac{-5}{6}} + \frac{6}{5} + \frac{5}{6}x^{-6} - 6x^{-5}\,dx$

$= \frac{5}{6}\,\frac{x^6}{6} + \frac{x^{\frac{11}{6}}}{\frac{11}{6}} + \frac{x^{\frac{1}{6}}}{\frac{1}{6}} + \frac{6}{5}x + \frac{5}{6}\left(\frac{x^{-5}}{-5}\right) \cancel{-} 6\frac{x^{-4}}{\cancel{-}4} + C$

$= \boxed{\frac{5}{36}x^6 + \frac{6}{11}x^{\frac{11}{6}} + 6x^{\frac{1}{6}} + \frac{6}{5}x - \frac{1}{6x^5} + \frac{3}{2x^4} + C}$

3(b) $\int \sec^2 x - 8\cos x + \sin x + \frac{\sec x\cdot\tan x}{7}\,dx$

$= \boxed{\tan x - 8\sin x - \cos x + \frac{1}{7}\cdot\sec x + C}$

3(c) $\int\left(x^2 + \frac{1}{x^2}\right)\left(x^2 - \frac{1}{x^2}\right)dx \overset{\text{FOIL}}{=} \int x^4 + \frac{\cancel{x^2}}{\cancel{x^2}}^{1} - \frac{\cancel{x^2}}{\cancel{x^2}}^{1} - \frac{1}{x^4}\,dx$

$= \int x^4 + \cancel{1} - \cancel{1} - \frac{1}{x^4}\,dx$

$= \int x^4 - x^{-4}\,dx$

$= \frac{x^5}{5} \cancel{-} \frac{x^{-3}}{\cancel{-}3} + C$

$= \boxed{\frac{x^5}{5} + \frac{1}{3x^3} + C}$

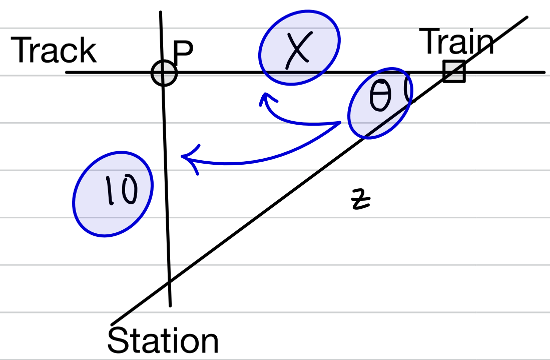
$$3(d)\ \int \frac{x^4 - 5x^2 + \sqrt{x} + 7}{x^2}\,dx \overset{\text{split}}{=} \int \frac{x^4}{x^2} - \cancel{\frac{5x^2}{x^2}} + \frac{\sqrt{x}}{x^2} + \frac{7}{x^2}\,dx$$

$$= \int x^2 - 5 + x^{-\frac{3}{2}} + 7x^{-2}\,dx$$

$$= \frac{x^3}{3} - 5x + \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + \frac{7x^{-1}}{-1} + c$$

$$= \boxed{\frac{x^3}{3} - 5x - \frac{2}{\sqrt{x}} - \frac{7}{x} + C}$$

4. Diagram → given



Variables

Let x = Distance between train and Point P

z = Distance between train and station

θ = Angle between track and line connecting the Train the Station

Given $\frac{dx}{dt} = 6$ ft/sec

Find $\frac{d\theta}{dt} = ?$ when $z = 20$

Equation

$$\tan\theta = \frac{10}{x}$$

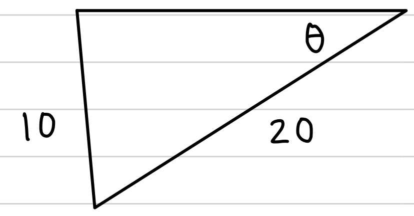
Differentiate

$$\frac{d}{dt}\left(\tan\theta\right) = \frac{d}{dt}\left(10x^{-1}\right)$$

$$(\sec\theta)^2 \quad \sec^2\theta \cdot \frac{d\theta}{dt} = -10x^{-2} \cdot \frac{dx}{dt}$$

Extra Solvable Information

$$? = \sqrt{(20)^2 - (10)^2} = \sqrt{400 - 100} = \sqrt{300} \overset{\text{OR}}{=} \sqrt{100}\sqrt{3} = 10\sqrt{3}$$



$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{A}{H}} = \frac{H}{A} = \frac{20}{\sqrt{300}}$$

Substitute

$$\left(\frac{20}{\sqrt{300}}\right)^2 \cdot \frac{d\theta}{dt} = -\frac{10}{(\sqrt{300})^2} \cdot 6$$

?

Solve $$\frac{d\theta}{dt} = -\frac{60}{3\cancel{00}} \cdot \frac{(\sqrt{\cancel{300}})^2}{(20)^2} \qquad \frac{\cancel{300}}{400}$$

$$= -\frac{60}{400} = -\frac{6}{40} = -\frac{3}{20}$$

Radians per Second

negative, decreasing makes sense

Answer $\boxed{\text{This angle is decreasing at a rate of } \frac{3}{20} \text{ Radians every second at this Moment.}}$

5. $f'(x) = \frac{32}{x^3} - \frac{1}{\sqrt{x}} + 2$ and $f(4) = -6$

Antidifferentiate $f(x) = \int f'(x)\,dx = \int \frac{32}{x^3} - \frac{1}{\sqrt{x}} + 2\,dx$

$$= \int 32x^{-3} - x^{-\frac{1}{2}} + 2\,dx$$

$$= \frac{\overset{16}{\cancel{32}}\,x^{-2}}{-\cancel{2}} - \frac{x^{1/2}}{1/2} + 2x + C$$

$$= -\frac{16}{x^2} - 2\sqrt{x} + 2x + C$$

Test Initial Condition

$$f(4) = \frac{-\cancel{16}}{\underset{16}{\cancel{4^2}}} - 2\overset{2}{\cancel{\sqrt{4}}} + 2\cdot 4 + C \overset{\text{set}}{=} -6$$

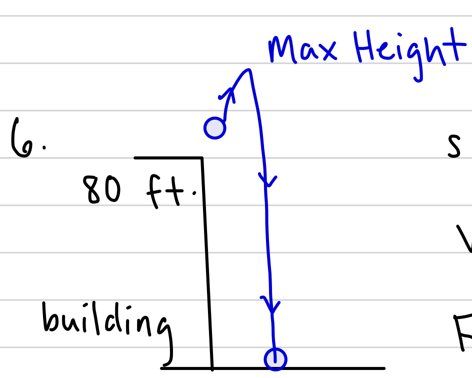
$$-1 - 4 + 8 + C = -6$$

$$3 + C = -6$$

$$\hookrightarrow C = -9$$

Finally, $\boxed{f(x) = \frac{-16}{x^2} - 2\sqrt{x} + 2x - 9}$

6.



$s(0) = 80$ ft.

$v(0) = 64$ ft/sec

Find $t_{max} = ?$

Find Max Height $\hookrightarrow s(t_{max}) = ?$

Find $t_{impact} = ?$

Find $v(t_{impact}) = ?$

$a(t) = -32$

$v(t) = -32t + \overset{64}{\cancel{v_0}}$

$= -32t + 64$

$s(t) = -16t^2 + \overset{64}{\cancel{v_0}}\,t + \overset{80}{\cancel{s_0}}$

$= -16t^2 + 64t + 80$

(a) Max Height is when $v(t)=0$

$v(t)=-32t+64 \overset{\text{set}}{=} 0$ ⤷ Solve $32t=64 \Rightarrow t_{max}=2$ sec.

Max Height occurs at $t=2$ seconds

(b) Max Height occurs when $t_{max}=2$

$\Rightarrow s(t_{max})=s(2)=-16(2)^2+64(2)+80=\overset{64}{\widehat{-64}}+128+80=144$ feet

$\boxed{\text{Max Height is 144 feet}}$

(c) Strikes ground when $s(t)=0$

$s(t)=-16t^2+64t+80=-16(t^2-4t-5)$

$=-16(t-5)(t+1)\overset{\text{set}}{=}0$

$t-5=0$ $\quad$ $t+1=0$

$t=5$ $\quad$ $\cancel{t=-1}$ ignore

$\boxed{\text{Hits ground after } t_{impact}=5 \text{ seconds}}$

(d) Velocity at impact $v(t_{impact})=v(5)=-32\cdot(5)+64=-160+64=-96$ ft/sec.

$\boxed{\text{Impact Velocity is } -96 \text{ feet per second}}$

negative makes sense. DOWN @ impact.