Answer Key

**Math 106** **Practice Exam #1** **Spring 2014**

**1.** Differentiate each of the following functions. **Do not** simplify your answers.

(a) $f(x) = \cos^4\left(\sin\left(\frac{7}{x}\right)\right)$

$$f'(x) = \boxed{4\cos^3\left(\sin\left(\frac{7}{x}\right)\right)\left(-\sin\left(\sin\left(\frac{7}{x}\right)\right)\right)\cos\left(\frac{7}{x}\right)(-7x^{-2})}$$

(b) $y = \sec^2(3x) \cdot \sec x$

$$y' = \boxed{\sec^2(3x)\sec x\tan x + \sec x(2\sec(3x))(3\sec(3x)\tan(3x))}$$

(c) $g(t) = \dfrac{t^3 + \tan\left(\frac{1}{t}\right)}{1+t^2}$

$$g'(t) = \boxed{\frac{(1+t^2)\left(3t^2 + \sec^2\left(\frac{1}{t}\right)\left(-\frac{1}{t^2}\right)\right) - \left(t^3 + \tan\left(\frac{1}{t}\right)\right)(2t)}{(1+t^2)^2}}$$

**2.** Consider the equation $\sec(x^2y) + \cos x = y^{\frac{3}{2}}$. Compute $\dfrac{dy}{dx}$.

First implicitly differentiate both sides:

$$\sec(x^2y)\tan(x^2y)\left(x^2\frac{dy}{dx} + y(2x)\right) - \sin x = \frac{3}{2}y^{\frac{1}{2}}\frac{dy}{dx}$$

$$x^2\sec(x^2y)\tan(x^2y)\frac{dy}{dx} + 2xy\sec(x^2y)\tan(x^2y) - \sin x = \frac{3}{2}\sqrt{y}\frac{dy}{dx}$$

Isolate $\dfrac{dy}{dx}$.

$$x^2\sec(x^2y)\tan(x^2y)\frac{dy}{dx} - \frac{3}{2}\sqrt{y}\frac{dy}{dx} = -2xy\sec(x^2y)\tan(x^2y) + \sin x$$

Factor:

$$\left(x^2\sec(x^2y)\tan(x^2y) - \frac{3}{2}\sqrt{y}\right)\frac{dy}{dx} = -2xy\sec(x^2y)\tan(x^2y) + \sin x$$

Solve:

$$\frac{dy}{dx} = \boxed{\frac{-2xy\sec(x^2y)\tan(x^2y) + \sin x}{x^2\sec(x^2y)\tan(x^2y) - \frac{3}{2}\sqrt{y}}}$$

**3.** Compute the equation of the tangent line for $\tan\left(\frac{x}{y}\right) + y = \sqrt{1-x}$. at the point $(0, 1)$.

Implicitly differentiate:

$$\sec^2\left(\frac{x}{y}\right)\left(\frac{y - x\frac{dy}{dx}}{y^2}\right) + \frac{dy}{dx} = \frac{1}{2\sqrt{1-x}}(-1)$$

Plug in the point values, $x = 0$ and $y = 1$

$$\sec^2 0\left(\frac{1-0}{1}\right) + \frac{dy}{dx} = -\frac{1}{2}$$

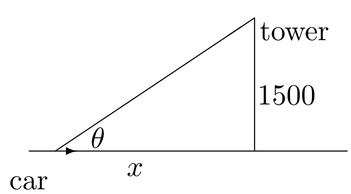
Finally, solve $\left.\frac{dy}{dx}\right|_{(0,1)} = -\frac{3}{2}$

Using point-slope form we get the equation of the tangent line:

$y - 1 = -\frac{3}{2}(x - 0)$ or $\boxed{y = -\frac{3}{2}x + 1}$

**4.** A child riding in a car driving along a straight road is looking through binoculars when she sees a water tower off to the side. The tower is located 1500 ft from the nearest point on the road. At a particular moment, the car is moving at 80 feet per second, and the car is 800 feet from that nearest point to the tower. How fast must the child be rotating the angle that the binoculars are pointing to keep the tower in view?

- Diagram




- Variables
  Let $x$ =distance from car to nearest point to tower at time $t$
  Let $\theta =$ angle binoculars form with straight road at time $t$
  Find $\frac{d\theta}{dt}$ =? when $x = 800$ feet
  and $\frac{dx}{dt} = -80\frac{\text{ft}}{\text{sec}}$
  Note this is negative b/c we are fixing driving left to right towards the nearest point.
- Equation relating the variables:

$$\tan\theta = \frac{1500}{x}$$

• Differentiate both sides w.r.t. time $t$.

$$\frac{d}{dt}(\tan\theta) = \frac{d}{dt}(\frac{1500}{x}) \implies \sec^2\theta \frac{d\theta}{dt} = -\frac{1500}{x^2}\frac{dx}{dt} \text{ (Related Rates!)}$$

• Extra Solvable Information

We compute that the hypotenuse is

$\sqrt{x^2+1500^2} = \sqrt{800^2+1500^2} = \sqrt{640,000+2250000} = 100\sqrt{289} = 1700$

so that at the key moment, $\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{1700}{800} = \frac{17}{8}$.

• Substitute Key Moment Information (now and not before now!!!):

We have $\left(\frac{17}{8}\right)^2 \frac{d\theta}{dt} = -\frac{1500}{(800)^2}(-80)$

• Solve for the desired quantity: So the rate of change of rotation is

$$\frac{d\theta}{dt} = \frac{-\frac{1500}{800^2}\cdot(-80)}{(17/8)^2} = \frac{12}{289} \text{ radians/sec} \cong 0.0415 \text{ radians/sec.}$$

• Answer the question that was asked: The child must be rotating the binoculars at a rate of $\frac{12}{289}$ radians every second.

**5.** Find a function $f$ such that $f''(x) = 2x^3+3x^2-4x+5$ and $f(0)=2$ and $f(1)=0$.

Antidifferentiate:

$f'(x) = \int f''(x)\ dx = \int 2x^3+3x^2-4x+5\ dx = \frac{1}{2}x^4+x^3-2x^2+5x+C_1$

which implies

$f(x) = \int f'(x)\ dx = \int \frac{1}{2}x^4+x^3-2x^2+5x+C_1\ dx = \frac{1}{10}x^5+\frac{1}{4}x^4-\frac{2}{3}x^3+\frac{5}{2}x^2+C_1x+C_2$

We will use the given Initial Conditions to solve for the Constants of antidifferentiation, $C_1$ and $C_2$.

First use the simpler condition $f(0)=2$

$f(0) = 0+0-0+0+0+C_2 \overset{\text{set}}{=} 2 \ \Rightarrow C_2 = 2$

Substitute for $C_2$ into $f(x)$

$f(x) = \frac{1}{10}x^5+\frac{1}{4}x^4-\frac{2}{3}x^3+\frac{5}{2}x^2+C_1x+2$

Next use $f(1) = 0$

$$f(1) = \frac{1}{10} + \frac{1}{4} - \frac{2}{3} + \frac{5}{2} + C_1 + 2 \overset{\text{set}}{=} 0$$

Using common denominators:

$$\frac{6}{60} + \frac{15}{60} - \frac{40}{60} + \frac{150}{60} + C_1 + \frac{120}{60} = 0$$

Then

$$\frac{251}{60} + C_1 = 0 \Rightarrow C_1 = -\frac{251}{60}$$

Finally,

$$f(x) = \boxed{\frac{1}{10}x^5 + \frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{5}{2}x^2 - \frac{251}{60}x + 2}$$

**6.** Find the general antiderivative for each of the following:

(a) $\dfrac{\sqrt{x} + \frac{1}{x}}{x^{\frac{3}{5}}}$

The most general antiderivative is given by

$$\int \frac{\sqrt{x} + \frac{1}{x}}{x^{\frac{3}{5}}} \, dx$$

First use algebra to Simplify to set yourself up for power rules.

$$\int \frac{\sqrt{x} + \frac{1}{x}}{x^{\frac{3}{5}}} \, dx = \int \frac{\sqrt{x}}{x^{\frac{3}{5}}} + \frac{\frac{1}{x}}{x^{\frac{3}{5}}} \, dx = \int \frac{x^{\frac{1}{2}}}{x^{\frac{3}{5}}} + \frac{x^{-1}}{x^{\frac{3}{5}}} \, dx$$

$$= \int x^{\frac{1}{2}-\frac{3}{5}} + x^{-1-\frac{3}{5}} \, dx = \int x^{\frac{5}{10}-\frac{6}{10}} + x^{-\frac{5}{5}-\frac{3}{5}} \, dx = \int x^{-\frac{1}{10}} + x^{-\frac{8}{5}} \, dx$$

Antidifferentiate:

$$= \frac{x^{\frac{9}{10}}}{\frac{9}{10}} + \frac{x^{-\frac{3}{5}}}{-\frac{3}{5}} + C = \boxed{\frac{10}{9}x^{\frac{9}{10}} - \frac{5}{3}x^{-\frac{3}{5}} + C}$$

(b) $\left(x^{\frac{3}{4}} + \dfrac{1}{\sqrt{x}}\right)\left(x - \dfrac{1}{x^{\frac{3}{4}}}\right)$

The most general antiderivative is given by

$$\int \left(x^{\frac{3}{4}} + \frac{1}{\sqrt{x}}\right)\left(x - \frac{1}{x^{\frac{3}{4}}}\right) \, dx$$

First Simplify to set yourself up for power rules.

$$\int \left(x^{\frac{3}{4}} + \frac{1}{\sqrt{x}}\right)\left(x - \frac{1}{x^{\frac{3}{4}}}\right)\, dx = \int \left(x^{\frac{3}{4}} + \frac{1}{\sqrt{x}}\right)\left(x - \frac{1}{x^{\frac{3}{4}}}\right)\, dx$$

$$= \int x^{\frac{7}{4}} + \frac{x}{\sqrt{x}} - \frac{x^{\frac{3}{4}}}{x^{\frac{3}{4}}} - \frac{1}{x^{\frac{1}{2}}x^{\frac{3}{4}}}\, dx$$

$$= \int x^{\frac{7}{4}} + x^{\frac{1}{2}} - 1 - x^{-\frac{5}{4}}\, dx$$

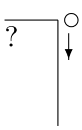
Antidifferentiate:

$$= \frac{x^{\frac{11}{4}}}{\frac{11}{4}} + \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - x - \frac{x^{-\frac{1}{4}}}{-\frac{1}{4}} + C = \boxed{\frac{4}{11}x^{\frac{11}{4}} + \frac{2}{3}x^{\frac{3}{2}} - x + 4x^{-\frac{1}{4}} + C}$$

**7.** Jack throws a baseball straight downward from the top of a building. The initial *speed* of the ball is 25 feet per second. It hits the ground with a speed of 153 feet per second. How tall is the building?

Hint: Use $a(t) = -32$ feet per second squared as acceleration due to gravity on the falling body.

Note $v_0 = -25\frac{\text{ft}}{\text{sec}}, s_0 = ?\text{ft}, v_{\text{impact}} = -153\frac{\text{ft}}{\text{sec}}$




$$a(t) = -32$$
$$v(t) = -32t + v_0 \Longrightarrow v(t) = -32t - 25$$
$$s(t) = -16t^2 - 25t + s_0$$

The ball hits the ground when $v(t) = -32t - 25 = -153$ or when $32t = 153 - 25 = 128$ which is when $t_{\text{impact}} = 4$ seconds.

Finally, we solve $s(4) = 0$ for $s_0$. The ball hits the ground when $-16(4)^2 - 25(4) + s_0 = 0$ or when $-256 - 100 + s_0 = 0$ which is when $s_0 = 356$ feet. As a result, the building is $\boxed{356}$ feet tall.