Extra Riemann Sum, Definite Integral Example

Evaluate $\displaystyle\int_1^5 5-2x-x^2\,dx$ using Riemann Sums and the limit definition of the definite integral.

Here $f(x) = 5 - 2x - x^2, \quad a = 1, \quad b = 5, \quad \Delta x = \dfrac{5-1}{n} = \dfrac{b-a}{n} = \dfrac{4}{n}$
and $x_i = a + i\Delta x = 1 + i\left(\dfrac{4}{n}\right) = 1 + \dfrac{4i}{n}$.

$$\begin{aligned}
\int_1^5 5 - 2x - x^2\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(1+\frac{4i}{n}\right)\frac{4}{n} \\
&= \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n} 5 - 2\left(1+\frac{4i}{n}\right) - \left(1+\frac{4i}{n}\right)^2 \\
&= \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n}\left(5 - 2 - \frac{8i}{n} - 1 - \frac{8i}{n} - \frac{16i^2}{n^2}\right) \\
&= \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n}\left(2 - \frac{16i}{n} - \frac{16i^2}{n^2}\right) \\
&= \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n} 2 - \frac{4}{n}\sum_{i=1}^{n}\frac{16i}{n} - \frac{4}{n}\sum_{i=1}^{n}\frac{16i^2}{n^2} \\
&= \lim_{n\to\infty}\frac{8}{n}\sum_{i=1}^{n} 1 - \frac{64}{n^2}\sum_{i=1}^{n} i - \frac{64}{n^3}\sum_{i=1}^{n} i^2 \\
&= \lim_{n\to\infty}\frac{8}{\cancel{n}}(\cancel{n}) - \frac{64}{n^2}\left(\frac{n(n+1)}{2}\right) - \frac{64}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) \\
&= \lim_{n\to\infty} 8 - \frac{64}{2}\left(\frac{\cancel{n}}{\cancel{n}}\right)\left(\frac{n+1}{n}\right) - \frac{64}{6}\left(\frac{\cancel{n}}{\cancel{n}}\right)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) \\
&= \lim_{n\to\infty} 8 - 32\,(1)\left(1 + \cancelto{0}{\frac{1}{n}}\right) - \frac{32}{3}\,(1)\left(1 + \cancelto{0}{\frac{1}{n}}\right)\left(2 + \cancelto{0}{\frac{1}{n}}\right) \\
&= 8 - 32(1)(1) - \frac{32}{3}(1)(1)(2) = -24 - \frac{64}{3} = -\frac{72}{3} - \frac{64}{3} = \boxed{-\frac{136}{3}}
\end{aligned}$$