Definite Integral Limit Definition using Riemann Sums

Definition: the **Definite Integral** of a function f from $x = a$ to $x = b$ is given by

$$(\bullet) \quad \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty} \left[f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \ldots + f(x_i)\Delta x + \ldots + f(x_n)\Delta x\right]$$

Note: The definite integral is a limit of a sum! Just think about this formula as

the **limiting value** *of the sum of the areas of finitely many (n) approximating rectangles.*

To compute definite integrals the *long (limit) way*, follow these steps:

Step 1: Given the integral $\int_a^b f(x)\,dx$, **pick off** or **identify** the **integrand** $f(x)$, and **limits of integration** a and b.

Step 2: Compute $\boxed{\Delta x = \frac{b-a}{n}}$. This width of each partitioned interval should be in terms of n.

Step 3: Compute $\boxed{x_i = a + i\Delta x}$. Leave the i as your counter. You have the left-most endpoint a from Step 1. You have width Δx from Step 2. This endpoint x_i should be in terms of i and n.

Step 4: Plug x_i and Δx into the formula $(\bullet)$ above. Here it is again:

$$(\bullet) \quad \boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x} \quad \longleftarrow \textbf{MEMORIZE!}$$

Step 5: Use the following formulas for sum of integers i and finish evaluating the limit in n.

$$\sum_{i=1}^{n} 1 = n$$

$$(*) \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$(**) \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$(***) \quad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Note: your final answer for the definite integral should be a **number** after you finish the limit.

1. **Read** through the entire next problem. Make sure you understand the formula to start, as well as the formulas for Δx and x_i. Because it doesn't feel natural yet, just trust the formulas right now.

Evaluate $\int_0^6 x^2\,dx$ using the Limit Definition of the Definite Integral using Riemann Sums.

Here $f(x) = x^2$, $a = 0,\ b = 6,\ \Delta x = \frac{b-a}{n} = \frac{6-0}{n} = \frac{6}{n}$ and $x_i = a+i\Delta x = 0+i\left(\frac{6}{n}\right) = \frac{6i}{n}$.

$$
\begin{aligned}
\int_0^6 x^2 dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x \quad &= \lim_{n\to\infty}\sum_{i=1}^{n} f\left(\frac{6i}{n}\right)\frac{6}{n} \\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\left(\frac{6i}{n}\right)^2\right)\frac{6}{n} \\
&= \lim_{n\to\infty}\frac{6}{n}\sum_{i=1}^{n}\frac{36i^2}{n^2} \\
&= \lim_{n\to\infty}\left(\frac{216}{n^3}\sum_{i=1}^{n} i^2\right) \\
&= \lim_{n\to\infty}\left(\frac{216}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right) \text{ using } (**) \\
&= \lim_{n\to\infty}\left(\frac{216}{6}\cdot\frac{n(n+1)(2n+1)}{n^3}\right) \\
&= \lim_{n\to\infty}\left(\frac{216}{6}\cdot\frac{n(n+1)(2n+1)}{n\cdot n\cdot n}\right) \\
&= \lim_{n\to\infty}\left(\frac{216}{6}\cdot\left(\frac{n}{n}\right)\cdot\left(\frac{n+1}{n}\right)\cdot\left(\frac{2n+1}{n}\right)\right) \\
&= \lim_{n\to\infty}\left(\frac{216}{6}\cdot 1\cdot\left(1+\frac{1}{n}\right)\cdot\left(2+\frac{1}{n}\right)\right) \\
&= \frac{216}{6}\cdot 1\cdot 2 = \frac{216}{3} = \boxed{72}
\end{aligned}
$$

2. **Read** through the entire next problem. Make sure you understand the formula to start, as well as the formulas for Δx and x_i. Here the lower limit of integration a is **not** 0.

Evaluate $\int_1^4 6 - 3x \; dx$ using the Limit Definition of the Definite Integral using Riemann Sums.

Here $f(x) = 6 - 3x, \; a = 1, \; b = 4, \; \Delta x = \dfrac{b-a}{n} = \dfrac{4-1}{n} = \dfrac{3}{n}$

and $x_i = a + i\Delta x = 1 + i\left(\dfrac{3}{n}\right) = 1 + \dfrac{3i}{n}$.

$$\begin{aligned}
\int_1^4 6 - 3x \; dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \quad &= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(1 + \frac{3i}{n}\right)\frac{3}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(6 - 3\left(1 + \frac{3i}{n}\right)\right)\frac{3}{n} \\
&= \lim_{n\to\infty} \left(\frac{3}{n}\sum_{i=1}^{n}\left(3 - \frac{9i}{n}\right)\right) \\
&= \lim_{n\to\infty} \left(\frac{3}{n}\left(\sum_{i=1}^{n} 3 - \sum_{i=1}^{n}\frac{9i}{n}\right)\right) \\
&= \lim_{n\to\infty} \left(\frac{9}{n}\sum_{i=1}^{n} 1 - \frac{27}{n^2}\sum_{i=1}^{n} i\right) \\
&= \lim_{n\to\infty} \left(\frac{9}{n}(n) - \frac{27}{n^2}\frac{n(n+1)}{2}\right) \text{ using } (*) \\
&= \lim_{n\to\infty} \left(9 - \frac{27}{2}\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right)\right) \\
&= \lim_{n\to\infty} \left(9 - \frac{27}{2}(1)\left(1 + \frac{1}{n}\right)\right) \\
&= 9 - \frac{27}{2} \\
&= \boxed{-\frac{9}{2}}
\end{aligned}$$