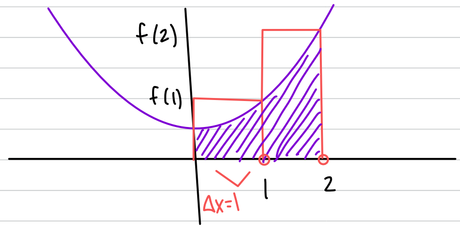
Worksheet 4 Answer Key

1. $f(x) = x^2 + 1$

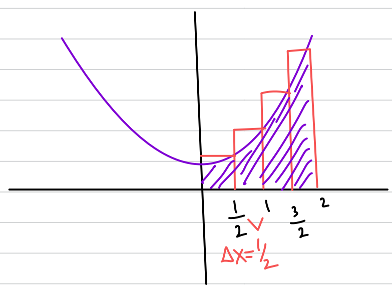


Δx width$=1$ here

$$\text{Area} \approx \text{Area Rectangle 1} + \text{Area Rectangle 2}$$
$$= \text{Height}_1 \cdot \text{Width} + \text{Height}_2 \cdot \text{Width}$$
$$= f(1)\cdot 1 + f(2)\cdot 1$$
$$= (1^2+1)\cdot 1 + (2^2+1)\cdot 1$$
$$= 2\cdot 1 + 5 \cdot 1 = 2 + 5 = 7$$

Overestimate

2 $f(x) = x^2 + 1$



Δx width $= \frac{1}{2}$ here

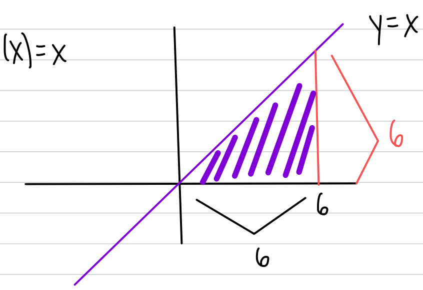
$$\text{Area} \approx \text{Area Rect}_1 + \text{Area Rect}_2 + \text{Area Rect}_3 + \text{Area Rect}_4$$
$$= H_1 \text{ Width} + H_2 \cdot \text{Width} + H_3 \text{ Width} + H_4 \cdot \text{Width}$$
$$= f\left(\tfrac{1}{2}\right)\cdot\tfrac{1}{2} + f(1)\cdot\tfrac{1}{2} + f\left(\tfrac{3}{2}\right)\tfrac{1}{2} + f(2)\cdot\tfrac{1}{2}$$
$$= \left(\left(\tfrac{1}{2}\right)^2+1\right)\tfrac{1}{2} + (1^2+1)\cdot\tfrac{1}{2} + \left(\underbrace{\left(\tfrac{3}{2}\right)^2+1}_{9/4+4/4}\right)\cdot\tfrac{1}{2} + (2^2+1)\cdot\tfrac{1}{2}$$
$$= \frac{5}{4}\cdot\frac{1}{2} + 2\,\frac{1}{2} + \frac{13}{4}\cdot\frac{1}{2} + 5\cdot\frac{1}{2}$$

OR, Can factor out $\frac{1}{2}$ ...

$$= \frac{5}{8} + \cancel{1}^{8/8} + \frac{13}{8} + \cancel{\frac{5}{2}}^{20/8} = \frac{5}{8} + \frac{8}{8} + \frac{13}{8} + \frac{20}{8} = \frac{46}{8}$$

Overestimate but better estimate

3. $f(x) = x$

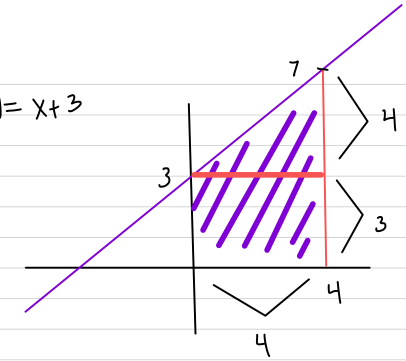


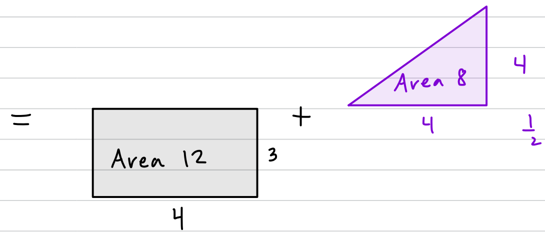
Area Interpretation

$$\text{Area} = \frac{1}{2}\text{ base}\cdot\text{height}$$
$$= \frac{1}{2}\cdot 6 \cdot 6 = \frac{36}{2} = 18$$

Function Value $f(6)=6$

4. $f(x) = x+3$

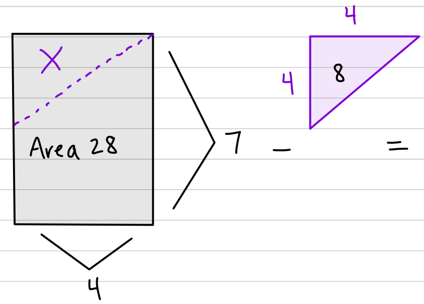




$\frac{1}{2}\, b \cdot h = \frac{1}{2}\; 4 \cdot 4 = \frac{16}{2} = 8$

Total Area $= 12 + 8 = 20$

OR//



$= 28 - 8 = 20$ ✓

5. $f(x) = x$

$$f\left(\frac{5i}{n}\right) = \frac{5i}{n} \quad \text{and} \quad f\left(2 + \frac{5i}{n}\right) = 2 + \frac{5i}{n}$$

$f(x)$ is the Identity here

6. $f(x) = 3x - 4$

$$f\left(\frac{8i}{n}\right) = 3\left(\frac{8i}{n}\right) - 4 = \frac{24i}{n} - 4$$

$$f\left(3 + \frac{8i}{n}\right) = 3\left(3 + \frac{8i}{n}\right) - 4 = 9 + \frac{24i}{n} - 4 = 5 + \frac{24i}{n}$$

7. $f(x) = x^2 + 5$

$$f\left(\frac{2i}{n}\right) = \left(\frac{2i}{n}\right)^2 + 5 = \frac{4i^2}{n^2} + 5$$

$$f\left(4 + \frac{2i}{n}\right) = \left(4 + \frac{2i}{n}\right)^2 + 5 = \left(4 + \frac{2i}{n}\right)\left(4 + \frac{2i}{n}\right) + 5$$

$$= 16 + \frac{8i}{n} + \frac{8i}{n} + \frac{4i^2}{n^2} + 5$$

$$= 21 + \frac{16i}{n} + \frac{4i^2}{n^2}$$

8. $f(x) = x^2 - 2x + 7$

$$f\left(\frac{6i}{n}\right) = \left(\frac{6i}{n}\right)^2 - 2\left(\frac{6i}{n}\right) + 7 = \frac{36i^2}{n^2} - \frac{12i}{n} + 7$$

$$f\left(-1 + \frac{6i}{n}\right) = \left(-1 + \frac{6i}{n}\right)^2 - 2\left(-1 + \frac{6i}{n}\right) + 7$$

$$= 1 - \frac{6i}{n} - \frac{6i}{n} + \frac{36i^2}{n^2} + 2 - \frac{12i}{n} + 7$$

$$= 10 - \frac{24i}{n} + \frac{36i^2}{n^2}$$

9 $\lim\limits_{n\to\infty} 3 = 3$ Limit of a constant equals the constant

10 $\lim\limits_{n\to\infty} \cancel{\frac{1}{n}}^{0} = 0$

11. $\lim\limits_{n\to\infty} 1 + \cancel{\frac{1}{n}}^{0} = 1$

12. $\lim\limits_{n\to\infty} \frac{n+1}{n} = \lim\limits_{n\to\infty} \cancel{\frac{n}{n}}^{1} + \cancel{\frac{1}{n}}^{0} = 1$

13 $\lim\limits_{n\to\infty} \frac{n+3}{n} = \lim\limits_{n\to\infty} \cancel{\frac{n}{n}}^{1} + \cancel{\frac{3}{n}}^{0} = 1$

14 $\lim\limits_{n\to\infty} \frac{2n+1}{n} = \lim\limits_{n\to\infty} \cancel{\frac{2n}{n}}^{2} + \cancel{\frac{1}{n}}^{0} = 2$

15. $\lim\limits_{n\to\infty} \frac{n(n+1)}{n^2} = \lim\limits_{n\to\infty} \left(\cancel{\frac{n}{n}}\right)\left(\frac{n+1}{n}\right) = \lim\limits_{n\to\infty} \cancel{\frac{n}{n}}^{1} + \cancel{\frac{1}{n}}^{0} = 1$ Partner

OR// $\lim\limits_{n\to\infty} \frac{n(n+1)}{n^2} = \lim\limits_{n\to\infty} \frac{n^2+n}{n^2} = \lim\limits_{n\to\infty} \cancel{\frac{n^2}{n^2}}^{1} + \frac{\cancel{n}}{n^{\cancel{2}\,1}} = \lim\limits_{n\to\infty} 1 + \cancel{\frac{1}{n}}^{0} = 1$

16. $\lim\limits_{n\to\infty} \frac{n(n+1)(2n+1)}{\underbrace{n^3}_{n\cdot n\cdot n}} = \lim\limits_{n\to\infty} \left(\cancel{\frac{n}{n}}^{1}\right)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) = \lim\limits_{n\to\infty} 1\cdot\left(1+\cancel{\frac{1}{n}}^{0}\right)\left(2+\cancel{\frac{1}{n}}^{0}\right)$

$= 1\cdot 1\cdot 2 = 2$

17. $\lim\limits_{n\to\infty} 3 - \left(\frac{4}{n^2}\right)\left(\frac{n(n+1)}{2}\right) - \left(\frac{12}{n^3}\right)\cdot\left(\frac{n(n+1)(2n+1)}{6}\right)$ Repartner

Split-Split

$= \lim\limits_{n\to\infty} 3 - \left(\cancel{\frac{4}{2}}^{2}\right)\left(\cancel{\frac{n}{n}}^{1}\right)\left(\frac{n+1}{n}\right) - \left(\cancel{\frac{12}{6}}^{2}\right)\left(\cancel{\frac{n}{n}}^{1}\right)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)$

$= 3 - 2\cdot 1 \left(\cancel{\frac{n}{n}}^{1} + \cancel{\frac{1}{n}}^{0}\right) - 2\cdot 1\cdot \left(\cancel{\frac{n}{n}}^{1} + \cancel{\frac{1}{n}}^{0}\right)\cdot\left(\cancel{\frac{2n}{n}}^{2} + \cancel{\frac{1}{n}}^{0}\right)$

$= 3 - 2\cdot 1\ 1 - 2\cdot 1\cdot 1\ 2 = 3-2-4 = -3$

18. $\sum\limits_{n=1}^{n} 6 = \underbrace{6+6+6+\cdots+6}_{n\text{ copies}} = 6n$

OR// $\sum\limits_{n=1}^{n} 6 = 6\sum\limits_{i=1}^{n} 1 = 6\underbrace{(1+1+1+\cdots+1)}_{n\text{ copies}} = 6\cdot n = 6n$

19. $$\sum_{i=1}^{n}(-3) = -3\underbrace{\sum_{i=1}^{n} 1}_{n} = -3\;n = -3n$$

20. $$\sum_{i=1}^{n}\left(\frac{6i}{n}-5\right)\cdot\left(\frac{6}{n}\right) = \sum_{i=1}^{n}\frac{36i}{n^2}-\frac{30}{n} = \sum_{i=1}^{n}\frac{36i}{n^2} - \sum_{i=1}^{n}\frac{30}{n}$$

$$= \frac{36}{n^2}\sum_{i=1}^{n} i - \frac{30}{n}\cancel{\sum_{i=1}^{n} 1}^{\,n}$$

$$= \frac{36}{n^2}\sum_{i=1}^{n} i - \cancel{\frac{30}{n}\cdot n} = \frac{36}{n^2}\sum_{i=1}^{n} i^2 - 30$$

Match

21. $$\sum_{i=1}^{n}\left(1+\frac{3i}{n}\right)^2\cdot\left(\frac{3}{n}\right) = \sum_{i=1}^{n}\left(1+\frac{6i}{n}+\frac{9i^2}{n^2}\right)\cdot\frac{3}{n}$$

$$= \sum_{i=1}^{n}\frac{3}{n}+\frac{18i}{n^2}+\frac{27i^2}{n^3}$$

$$= \sum_{i=1}^{n}\frac{3}{n} + \sum_{i=1}^{n}\frac{18i}{n^2} + \sum_{i=1}^{n}\frac{27i^2}{n^3}$$

$$= \frac{3}{\cancel{n}}\cancel{\sum_{i=1}^{n} 1}^{\,\cancel{n}} + \frac{18}{n^2}\sum_{i=1}^{n} i + \frac{27}{n^3}\sum_{i=1}^{n} i^2$$

$$= 3 + \frac{18}{n^2}\sum_{i=1}^{n} i + \frac{27}{n^3}\sum_{i=1}^{n} i^2$$