Math 106, Spring 2018

Worksheet 12, Tuesday, April 17th, 2018

Derivatives Compute each of the following derivatives.

1. $f'(x)$ where $f(x) = \dfrac{1}{\sqrt{\ln x}} + \dfrac{1}{\ln \sqrt{x}} + e^{\sqrt{\ln x}} + \dfrac{1}{e^{\sqrt{x}}} + \dfrac{1}{e^x + \ln x} + \dfrac{e^x}{\ln x}$ Do not simplify.

   First rewrite $f(x) = (\ln x)^{-\frac{1}{2}} + (\ln \sqrt{x})^{-1} + e^{\sqrt{\ln x}} + e^{-\sqrt{x}} + (e^x + \ln x)^{-1} + \dfrac{e^x}{\ln x}$

   Differentiate:

   $$f'(x) = \boxed{-\tfrac{1}{2}(\ln x)^{-\frac{3}{2}}\left(\frac{1}{x}\right) - (\ln \sqrt{x})^{-2}\left(\frac{1}{\sqrt{x}}\right)\left(\frac{1}{2\sqrt{x}}\right) + e^{\sqrt{\ln x}}\left(\frac{1}{2\sqrt{\ln x}}\right)\left(\frac{1}{x}\right)}$$

   $$\text{(continued)}\boxed{\ldots + e^{-\sqrt{x}}\left(-\frac{1}{2\sqrt{x}}\right) - (e^x + \ln x)^{-2}\left(e^x + \frac{1}{x}\right) + \frac{\ln x(e^x) - e^x\left(\frac{1}{x}\right)}{(\ln x)^2}}$$

2. $\dfrac{d}{dx}(\sec x)^x$

   Let $y = (\sec x)^x$

   Then $\ln y = \ln\left((\sec x)^x\right) = x \ln(\sec x)$

   Implicitly differentiate

   $$\frac{1}{y}\frac{dy}{dx} = x\left(\frac{1}{\sec x}\right)\sec x \tan x + \ln(\sec x)(1)$$

   Solve $\dfrac{dy}{dx} = y\left[x \tan x + \ln(\sec x)\right] = \boxed{(\sec x)^x\left[x \tan x + \ln(\sec x)\right]}$

3. $\dfrac{dy}{dx}$, if $ye^{\ln y} + \sin^2 x = \ln 5 + e^{xy}$.

   First notice that $e^{\ln y} = y$, so the equation simplifies to

   $y^2 + \sin^2 x = \ln 5 + e^{xy}$

   Implicity differentiate both sides with respect to x

   $$\frac{d}{dx}\left(y^2 + \sin^2 x\right) = \frac{d}{dx}\left(\ln 5 + e^{xy}\right)$$

   $$2y\frac{dy}{dx} + 2\sin x(\cos x) = 0 + e^{xy}\left(x\frac{dy}{dx} + y(1)\right)$$

   Distribute

   $$2y\frac{dy}{dx} + 2\sin x(\cos x) = xe^{xy}\frac{dy}{dx} + ye^{xy}$$

Isolate $\frac{dy}{dx}$

$2y\frac{dy}{dx} - xe^{xy}\frac{dy}{dx} = ye^{xy} - 2\sin x(\cos x)$

Factor $\frac{dy}{dx}$

$(2y - xe^{xy})\frac{dy}{dx} = ye^{xy} - 2\sin x(\cos x)$

Solve

$$\frac{dy}{dx} = \boxed{\frac{ye^{xy} - 2\sin x(\cos x)}{2y - xe^{xy}}}$$

4. y' where $y = \left(\frac{\sqrt{1-x} \quad e^{\sec^2 x}}{(x\sin x)^{\frac{6}{7}}}\right)$ Do not simplify the final answer.

We will attack both sides with natural log, to simplify using logarithmic algebra

$$\ln y = \ln\left(\frac{\sqrt{1-x} \quad e^{\sec^2 x}}{(x\sin x)^{\frac{6}{7}}}\right)$$

$$\ln y = \ln\left(\sqrt{1-x} \quad e^{\sec^2 x}\right) - \ln\left((x\sin x)^{\frac{6}{7}}\right)$$

$$\ln y = \ln\left((1-x)^{\frac{1}{2}}\right) + \ln e^{\sec^2 x} - \frac{6}{7}\ln(x\sin x)$$

$$\ln y = \ln\left((1-x)^{\frac{1}{2}}\right) + \ln e^{\sec^2 x} - \frac{6}{7}(\ln x + \ln\sin x)$$

$$\ln y = \frac{1}{2}\ln(1-x) + \sec^2 x - \frac{6}{7}(\ln x + \ln\sin x)$$

Implicitly differentiate

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left(\frac{1}{1-x}\right)(-1) + 2\sec x(\sec x\tan x) - \frac{6}{7}\left(\frac{1}{x} + \left(\frac{1}{\sin x}\right)\cos x\right)$$

Solve

$$\frac{dy}{dx} = y\left[-\left(\frac{1}{2(1-x)}\right) + 2\sec^2 x\tan x - \frac{6}{7}\left(\frac{1}{x} + \left(\frac{1}{\sin x}\right)\cos x\right)\right]$$

Resubstitute

$$\frac{dy}{dx} = \boxed{\left(\frac{\sqrt{1-x} \quad e^{\sec^2 x}}{(x\sin x)^{\frac{6}{7}}}\right)\left[-\left(\frac{1}{2(1-x)}\right) + 2\sec^2 x\tan x - \frac{6}{7}\left(\frac{1}{x} + \left(\frac{1}{\sin x}\right)\cos x\right)\right]}$$

yikes!

Integration Compute each of the following integrals:

5. $\displaystyle\int \frac{(x\sqrt{x}-5)(1+\sqrt{x})}{x^3}\,dx = \int \frac{(x^{\frac{3}{2}}-5)(1+x^{\frac{1}{2}})}{x^3}\,dx = \int \frac{x^{\frac{3}{2}}+x^2-5-5x^{\frac{1}{2}}}{x^3}\,dx$

$\displaystyle = \int x^{-\frac{3}{2}} + \frac{1}{x} - 5x^{-3} - 5x^{-\frac{5}{2}}\,dx = \boxed{-2x^{-\frac{1}{2}} + \ln|x| + \frac{5}{2}x^{-2} + \frac{10}{3}x^{-\frac{3}{2}} + C}$

6. $\displaystyle\int_{e^3}^{e^8} \frac{8}{x\sqrt{1+\ln x}}\,dx = 8\int_4^9 \frac{1}{\sqrt{u}}\,du = 16\sqrt{u}\Big|_4^9 = 16(\sqrt{9}-\sqrt{4} = 16(3-2) = \boxed{16}$

$$\boxed{\begin{aligned} u &= 1+\ln x \\ du &= \frac{1}{x}dx \end{aligned}}$$

$$\boxed{\begin{aligned} x = e^3 &\quad \Rightarrow u = 1+\ln e^3 = 1+3 = 4 \\ x = e^8 &\quad \Rightarrow u = 1+\ln e^8 = 1+8 = 9 \end{aligned}}$$

7. $\displaystyle\int_0^{\ln 2} \frac{e^{2x}-e^{-2x}}{(e^x+e^{-x})^2}\,dx$ hint: try a combination of algebra **and** u-substitution

$\displaystyle\int_0^{\ln 2} \frac{e^{2x}-e^{-2x}}{(e^x+e^{-x})^2}\,dx = \int_0^{\ln 2} \frac{e^{2x}-e^{-2x}}{e^{2x}+2+e^{-2x}}\,dx = \frac{1}{2}\int_4^{\frac{25}{4}} \frac{1}{u}\,du$

$\displaystyle = \frac{1}{2}\ln|u|\Big|_4^{\frac{25}{4}} = \frac{1}{2}\left(\ln\left(\frac{25}{4}\right) - \ln 4\right) = \frac{1}{2}\ln\left(\frac{\frac{25}{4}}{4}\right) = \frac{1}{2}\ln\left(\frac{25}{16}\right)$

$\displaystyle = \ln\left(\sqrt{\frac{25}{16}}\right) = \boxed{\ln\left(\frac{5}{4}\right)}$

$$\boxed{\begin{aligned} u &= e^{2x}+2+e^{-2x} \\ du &= 2e^{2x}-2e^{-2x}dx \\ \frac{1}{2}du &= e^{2x}-e^{-2x}dx \end{aligned}}$$

$$\boxed{\begin{aligned} x = 0 &\quad \Rightarrow u = e^0+2+e^0 = 1+2+1 = 4 \\ x = \ln 2 &\quad \Rightarrow u = e^{2\ln 2}+2+e^{-2\ln 2} = e^{\ln(2^2)}+2+e^{\ln(2^{-2})} = 4+2+\frac{1}{4} = \frac{25}{4} \end{aligned}}$$

8. $\displaystyle\int_0^{\frac{\pi}{6}} \frac{\sec^2(2x)}{1+\tan(2x)}\,dx = \frac{1}{2}\int_1^{1+\sqrt{3}} \frac{1}{u}\,du = \frac{1}{2}\ln|u|\Big|_1^{1+\sqrt{3}} = \frac{1}{2}\left(\ln(1+\sqrt{3}) - \ln 1\right)$

$\displaystyle = \boxed{\frac{1}{2}\ln(1+\sqrt{3})}$

$$\boxed{\begin{aligned} u &= 1+\tan(2x) \\ du &= 2\sec^2(2x)dx \\ \frac{1}{2}du &= \sec^2(2x)dx \end{aligned}}$$

$$\boxed{\begin{aligned} x = 0 &\quad \Rightarrow u = 1+\tan 0 = 1+0 = 1 \\ x = \frac{\pi}{6} &\quad \Rightarrow u = 1+\tan\left(\frac{\pi}{3}\right) = 1+\sqrt{3} \end{aligned}}$$

9. $\displaystyle\int_0^{\frac{\pi}{3}} \frac{\sin x}{\cos^2 x\, e^{\sec x}}\ dx = \int_1^2 \frac{1}{e^u}\ du = \int_1^2 e^{-u}\ du - e^{-u}\Big|_1^2 = -e^{-2} + e^{-1} = \boxed{\frac{1}{e} - \frac{1}{e^2}}$

$$\boxed{\begin{aligned} u &= \sec x \\ du &= \sec x \tan x = \left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{\cos x}\right) dx = \left(\frac{\sin x}{\cos^2 x}\right) dx \end{aligned}}$$

$$\boxed{\begin{aligned} x = 0 &\quad \Rightarrow u = \sec 0 = 1 \\ x = \frac{\pi}{3} &\quad \Rightarrow u = \sec\left(\frac{\pi}{3}\right) = \frac{1}{\cos\left(\frac{\pi}{3}\right)} = \frac{1}{\left(\frac{1}{2}\right)} = 2 \end{aligned}}$$

10. Find the equation of the tangent line to

$$y = \ln(1+\cos x) - e\cos(\ln(1+x)) + e^{1+\ln(1+x)} + (\sin x)e^{\cos x}$$

at the point where the x-coordinate is 0.

First find the y-coordinate

$y = \ln(1+\cos 0) - e\cos(\ln(1+0)) + e^{1+\ln(1+0)} + (\sin 0)e^{\cos 0}$

$= \ln(1+1) - e\cos(\ln(1)) + e^{1+\ln(1)} + (0)e^1 = \ln 2 - e\cos(0) + e^{1+0} + 0$

$= \ln 2 - e(1) + e = \ln 2 - e + e = \ln 2$

Second compute

$y' = \dfrac{1}{1+\cos x}(-\sin x) + e\sin(\ln(1+x))\left(\dfrac{1}{1+x}\right) + e^{1+\ln(x+1)}\left(\dfrac{1}{x+1}\right)$

(continued)... $\quad + \sin x e^{\cos x}(-\sin x) + e^{\cos x}\cos x$

$y'(0) = \dfrac{1}{1+\cos 0}(-\sin 0) + e\sin(\ln(1+0))\left(\dfrac{1}{1+0}\right) + e^{1+\ln(0+1)}\left(\dfrac{1}{0+1}\right)$

(continued)... $\quad + \sin 0 e^{\cos 0}(-\sin 0) + e^{\cos 0}\cos 0 = 0 + 0 + e(1) + e(1) = 2e$

using Point-Slope form

$y - \ln 2 = 2e(x-0)$

Finally, $y = \boxed{2ex + \ln 2}$

Turn in your own solutions.