Math 106 Worksheet #11 Answer Key

1. Prove $\frac{d}{dx}(\ln x) = \frac{1}{x}$

Step 1: Let $y = \ln x$

Step 2: Invert

$$e^y = \cancel{e}^{\cancel{\ln} x}$$

$$e^y = x$$

Step 3: Implicitly Differentiate

$$\frac{d}{dx}\left[e^y\right] = \frac{d}{dx}\left[x\right]$$

$$e^y \cdot \frac{dy}{dx} = 1$$

Step 4: Solve

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x} \quad ✓$$

OR $$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{\cancel{e}^{\ln x}} = \frac{1}{x} \quad ✓$$

2. Let $y = x^{\sin x}$.

$$\ln y = \ln\left[x^{\sin x}\right]$$

$$\ln y = \sin x \cdot \ln x$$

Differentiate:

$$\frac{d}{dx}\left[\ln y\right] = \frac{d}{dx}\left[\sin x \cdot \ln x\right]$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \sin x \cdot \frac{1}{x} + \ln x \cdot \cos x$$

Solve: $$\frac{dy}{dx} = \cancel{y}^{\,x^{\sin x}}\left[\frac{\sin x}{x} + \ln x \cdot \cos x\right]$$

$$= \boxed{x^{\sin x}\left[\frac{\sin x}{x} + \ln x \cdot \cos x\right]}$$

3. $$\frac{d}{dx}\left[\ln\left(\frac{x^{3/4}\sqrt{x^2+1}}{e^{\sec x}}\right)\right] = \frac{d}{dx}\left[\ln\left(x^{3/4}\sqrt{x^2+1}\right) - \cancel{\ln e}^{\sec x}\right]$$

use Algebra First to Simplify

$$= \frac{d}{dx}\left[\underbrace{\ln(x^{3/4})}_{\frac{3}{4}\ln x} + \underbrace{\ln\sqrt{x^2+1}}_{\ln\left[(x^2+1)^{1/2}\right]} - \sec x\right]$$

$$= \frac{d}{dx}\left[\frac{3}{4}\ln x + \frac{1}{2}\ln(x^2+1) - \sec x\right]$$

$$= \frac{3}{4x} + \frac{1}{\cancel{2}}\left(\frac{1}{x^2+1}\right)(\cancel{2}x) - \sec x \tan x = \boxed{\frac{3}{4x} + \frac{x}{x^2+1} - \sec x \tan x}$$

4. $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{1}{u}\,du = -\ln|u| + C = \boxed{-\ln|\cos x| + C}$

$$\boxed{\begin{aligned} u &= \cos x \\ du &= -\sin x\,dx \\ -du &= \sin x\,dx \end{aligned}}$$

5. $\int_0^{\ln 2} \frac{e^{3x}}{\sqrt{8+e^{3x}}}\,dx = \frac{1}{3}\int_9^{16} \underbrace{\frac{1}{\sqrt{u}}}^{u^{-1/2}}\,du = \frac{1}{3}\cdot 2u^{1/2}\Big|_9^{16} = \frac{2}{3}\sqrt{u}\Big|_9^{16}$

$= \frac{2}{3}\left[\sqrt{16}^{\,4} - \sqrt{9}^{\,3}\right]$

$= \frac{2}{3}\left[\cancel{4-3}^{\,1}\right] = \boxed{\frac{2}{3}}$

$$\boxed{\begin{aligned} u &= 8+e^{3x} \\ du &= 3e^{3x}\,dx \\ \tfrac{1}{3}\,du &= e^{3x}\,dx \end{aligned}}$$

$$\boxed{\begin{aligned} &x=0 \Rightarrow u = 8+\cancel{e^{0}}^{\,1} = 9 \\ &x=\ln 2 \Rightarrow u = 8+e^{3\ln 2} \\ &\qquad = 8+\cancel{e^{\ln(2^3)}}^{\,8} = 16 \end{aligned}}$$

6. $\int_0^1 \frac{e^x}{2+e^x}\,dx = \int_3^{2+e} \frac{1}{u}\,du = \ln|u|\Big|_3^{2+e} = \boxed{\ln|2+e| - \ln|3|}$

$= \boxed{\ln\left[\frac{2+e}{3}\right]}$

can drop $|\cdot|$, all $\oplus$

$$\boxed{\begin{aligned} u &= 2+e^x \\ du &= e^x\,dx \end{aligned}}$$

$$\boxed{\begin{aligned} &x=0 \Rightarrow u = 2+\cancel{e^{0}}^{\,1} = 3 \\ &x=1 \Rightarrow u = 2+e \end{aligned}}$$

7. $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \boxed{\ln|e^x + e^{-x}| + C}$

$$\boxed{\begin{aligned} u &= e^x + e^{-x} \\ du &= e^x - e^{-x}\,dx \end{aligned}}$$

8. $\int_e^{e^4} \frac{3}{x\sqrt{\ln x}}\,dx = 3\int_1^4 \frac{1}{\sqrt{u}}\,du = 3\cdot 2u^{1/2}\Big|_1^4 = 6\sqrt{u}\Big|_1^4$

$u^{-1/2}$

$= 6\left[\sqrt{4} - \sqrt{1}\right] = 6(2-1) = \boxed{6}$

$\boxed{\begin{array}{l} u = \ln x \\ du = \frac{1}{x}\,dx \end{array}}$ $\boxed{\begin{array}{l} x = e \Rightarrow u = \ln e = 1 \\ x = e^4 \Rightarrow u = \ln e^4 = 4 \end{array}}$

9. $\int \frac{1+x^3}{x^4}\,dx \underset{\text{split}}{=} \int \frac{1}{x^4} + \frac{x^3}{x^4}\,dx = \int x^{-4} + \frac{1}{x}\,dx$

power rule, log

$= \frac{x^{-3}}{-3} + \ln|x| + C = \boxed{\frac{-1}{3x^3} + \ln|x| + C}$

10. $\int \frac{1}{e^x(1+e^{-x})}\,dx = -\int \frac{1}{u}\,du = -\ln|u| + C = \boxed{-\ln\left|1+e^{-x}\right| + C}$

$\boxed{\begin{array}{l} u = 1+e^{-x} \\ du = -e^{-x}\,dx \\ -du = \frac{1}{e^x}\,dx \end{array}}$

11. $\displaystyle\int_e^{e^5} \frac{1}{x\ln x}\,dx = \int_1^5 \frac{1}{u}\,du = \ln|u|\Big|_1^5 = \ln 5 - \cancel{\ln 1}^{\,0} = \boxed{\ln 5}$

$\boxed{\begin{array}{l} u=\ln x \\ du = \frac{1}{x}\,dx \end{array}}$ $\boxed{\begin{array}{l} x=e \Rightarrow u=\ln e = 1 \\ x=e^5 \Rightarrow u=\ln e^5 = 5 \end{array}}$

OR

$\displaystyle\int_e^{e^5} \frac{1}{x\ln x}\,dx = \int_{x=e}^{x=e^5} \frac{1}{u}\,du = \ln|u|\Big|_{x=e}^{x=e^5} = \ln|\ln x|\Big|_e^{e^5} = \ln|\cancel{\ln e^5}^{\,5}| - \ln|\cancel{\ln e}^{\,1}|$

$= \ln 5 - 0$

$= \boxed{\ln 5}$

0

12. $y = \sin(e^x)$ $\quad y(\ln\pi) = \sin\left(e^{\cancel{\ln \pi}^{\,\pi}}\right) = \sin \pi = 0$

Point $(\ln\pi, 0)$

Derivative

$y' = \cos(e^x)\cdot e^x$

Slope $y'(\ln\pi) = \cancel{\cos\left(e^{\cancel{\ln\pi}^{\,\pi}}\right)}^{\,-1} \cdot e^{\cancel{\ln\pi}^{\,\pi}} = -\pi$

Point-Slope Form:

$y - 0 = -\pi(x - \ln\pi)$

$\boxed{y = -\pi x + \pi\ln\pi}$