Math 106, Spring 2018

Worksheet 6, Tuesday, February 28th, 2018

1. Compute $\int_2^5 x^2\,dx$ using each of the following two methods:

   (a) The Fundamental Theorem of Calculus.

   $$\int_2^5 x^2\,dx = \frac{x^3}{3}\Big|_2^5 = \frac{5^3}{3} - \frac{2^3}{3} = \frac{125}{3} - \frac{8}{3} = \frac{117}{3} = \boxed{39}$$

   (b) The *Limit Definition* of the Definite Integral

   Here $f(x) = x^2$, $a = 2$, $b = 5$, $\Delta x = \frac{5-2}{n} = \frac{3}{n}$, $x_i = a + x_i = 2 + i\left(\frac{3}{n}\right) = 2 + \frac{3i}{n}$.

$$\begin{aligned}
\int_2^5 x^2\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(2 + \frac{3i}{n}\right)\left(\frac{3}{n}\right) \\
&= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left(2 + \frac{3i}{n}\right)^2 \\
&= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} 4 + \frac{12i}{n} + \frac{9i^2}{n^2} \\
&= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} 4 + \frac{3}{n} \sum_{i=1}^{n} \frac{12i}{n} + \frac{3}{n} \sum_{i=1}^{n} \frac{9i^2}{n^2} \\
&= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} 4 + \frac{36}{n^2} \sum_{i=1}^{n} i + \frac{27}{n^3} \sum_{i=1}^{n} i^2 \\
&= \lim_{n\to\infty} \frac{12}{n}(n) + \frac{36}{n^2}\left(\frac{n(n+1)}{2}\right) + \frac{27}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) \\
&= \lim_{n\to\infty} 12 + \frac{36}{2}\left(\frac{n(n+1)}{n^2}\right) + \frac{27}{6}\left(\frac{n(n+1)(2n+1)}{n^3}\right) \\
&= \lim_{n\to\infty} 12 + \frac{36}{2}\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right) + \frac{27}{6}\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) \\
&= \lim_{n\to\infty} 12 + 18(1)\left(1 + \frac{1}{n}\right) + \frac{27}{6}(1)\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) \\
&= 12 + 18(1)(1) + \frac{27}{6}(1)(1)(2) = 12 + 18 + \frac{54}{6} = 12 + 18 + 9 = \boxed{39}
\end{aligned}$$

2. Compute $f'(x)$ where $f(x) = \int_5^x \frac{1}{t+7}\, dt$.

using FTC Part I $f'(x) = \frac{d}{dx}\int_5^x \frac{1}{t+7}\, dt = \boxed{\frac{1}{x+7}}$

3. Compute $f'(x)$ where $f(x) = \int_x^9 \sqrt{t^2+3}\, dt$.

using FTC Part I $f'(x) = \frac{d}{dx}\int_x^9 \sqrt{t^2+3}\, dt = \frac{d}{dx}\left(-\int_9^x \sqrt{t^2+3}\, dt\right) = \boxed{-\sqrt{x^2+3}}$

4. Compute $g''(x)$ where $g(x) = \int_x^9 \sqrt{1+\cos t}\, dt$.

First compute $g'(x) = \frac{d}{dx}\int_x^9 \sqrt{1+\cos t}\, dt = \frac{d}{dx}\left(-\int_9^x \sqrt{1+\cos t}\, dt\right) = -\sqrt{1+\cos x}$

Next, $g''(x) = \frac{d}{dx}\left(-\sqrt{1+\cos x}\right) = -\frac{1}{2\sqrt{1+\cos x}}(-\sin x) = \boxed{\frac{\sin x}{2\sqrt{1+\cos x}}}$

NOTE: Unless instructions specify to use the Limit Definition of the Definite Integral, you may use the Fundamental Theorem of Calculus, Part II.

5. Compute $\int_0^{\frac{\pi}{3}} \sec^2\theta\, d\theta = \tan\theta\Big|_0^{\frac{\pi}{3}} = \tan\left(\frac{\pi}{3}\right) - \tan 0 = \sqrt{3} - 0 = \boxed{\sqrt{3}}$.

6. Compute $\int_{-\pi}^{\frac{\pi}{3}} \cos x\, dx = \sin x\Big|_{-\pi}^{\frac{\pi}{3}} = \sin\left(\frac{\pi}{3}\right) - \sin(-\pi) = \frac{\sqrt{3}}{2} - 0 = \boxed{\frac{\sqrt{3}}{2}}$.

7. Compute $\int_{-2}^{-1} x - \frac{5}{x^3}\, dx = \int_{-2}^{-1} x - 5x^{-3}\, dx = \frac{x^2}{2} - \frac{5x^{-2}}{-2}\Big|_{-2}^{-1} = \frac{x^2}{2} + \frac{5}{2x^2}\Big|_{-2}^{-1}$

$= \frac{1}{2} + \frac{5}{2} - \left(2 + \frac{5}{8}\right) = 3 - 2 - \frac{5}{8} = 1 - \frac{5}{8} = \boxed{\frac{3}{8}}$.

8. Compute $\int_0^{\frac{\pi}{6}} (\tan x + \sec x)\sec x\, dx = \int_0^{\frac{\pi}{6}} \sec x\tan x + \sec^2 x\, dx = \sec x + \tan x\Big|_0^{\frac{\pi}{6}}$

$= \sec\left(\frac{\pi}{6}\right) + \tan\left(\frac{\pi}{6}\right) - (\sec 0 + \tan 0) = \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} - (1+0) = \frac{3}{\sqrt{3}} - 1 = \boxed{\sqrt{3}-1}$.

9. Compute $\int_1^2 \left(x^2 - \frac{1}{x^2}\right)^2 dx = \int_1^2 x^4 - 2 + \frac{1}{x^4}\, dx = \frac{x^5}{5} - 2x + \frac{x^{-3}}{-3}\Big|_1^2$

$= \frac{2^5}{5} - 4 - \frac{2^{-3}}{3} - \left(\frac{1}{5} - 2 - \frac{1}{3}\right) = \frac{32}{5} - 4 - \frac{1}{24} - \frac{1}{5} + 2 + \frac{1}{3}$

$= \frac{31}{5} - 2 + \frac{7}{24} = \frac{744}{120} - \frac{240}{120} + \frac{35}{120} = \boxed{\frac{539}{120}}$.

10. Compute $\int_0^1 x^{\frac{3}{4}} - 2x^{\frac{1}{2}}\, dx = \frac{x^{\frac{7}{4}}}{\frac{7}{4}} - 2\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\Big|_0^1 = \frac{4}{7}x^{\frac{7}{4}} - \frac{4}{3}x^{\frac{3}{2}}\Big|_0^1 = \frac{4}{7} - \frac{4}{3} - (0-0) = \frac{12}{21} - \frac{28}{21} = \boxed{-\frac{16}{21}}$.

11. Compute $\int_1^4 \frac{x - x^3}{\sqrt{x}}\, dx = \int_1^4 \frac{x}{\sqrt{x}} - \frac{x^3}{\sqrt{x}}\, dx = \int_1^4 x^{\frac{1}{2}} - x^{\frac{5}{2}}\, dx = \frac{2}{3}x^{\frac{3}{2}} - \frac{2}{7}x^{\frac{7}{2}}\Big|_1^4$

$= \frac{2}{3}4^{\frac{3}{2}} - \frac{2}{7}4^{\frac{7}{2}} - \left(\frac{2}{3} - \frac{2}{7}\right) = \frac{2}{3}(\sqrt{4})^3 - \frac{2}{7}(\sqrt{4})^7 - \frac{2}{3} + \frac{2}{7}$

$= \frac{2}{3}(8) - \frac{2}{7}(128) - \frac{2}{3} + \frac{2}{7} = \frac{16}{3} - \frac{256}{7} - \frac{2}{3} + \frac{2}{7} = \frac{14}{3} - \frac{254}{7} = \frac{98}{21} - \frac{762}{21} = \boxed{-\frac{664}{21}}.$

12. Compute $\int_{-2}^{1} |x|\, dx$. Recall how the absolute value is defined. Then draw the bounded region and use *area interpretation* to confirm your answer.

Recall $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

$\int_{-2}^{1} |x|\, dx = \int_{-2}^{0} -x\, dx + \int_0^1 x\, dx = -\frac{x^2}{2}\Big|_{-2}^{0} + \frac{x^2}{2}\Big|_0^1 = 2 + \frac{1}{2} = \boxed{\frac{5}{2}}$

*See me for a diagram. Compute the area bounded above the x-axis and subtract off the area bounded below the x-axis.

13. Compute $\int_4^7 |x - 5|\, dx$. Again, draw the bounded region and use *area interpretation* to confirm your answer.

Recall $|x - 5| = \begin{cases} x - 5 & \text{if } x \geq 5 \\ -(x - 5) & \text{if } x < 5 \end{cases}$

$\int_4^7 |x - 5|\, dx = \int_4^5 |x - 5|\, dx + \int_5^7 |x - 5|\, dx = \int_4^5 -(x - 5)\, dx + \int_5^7 x - 5\, dx$

$= \int_4^5 -x + 5\, dx + \int_5^7 x - 5\, dx = -\frac{x^2}{2} + 5x\Big|_4^5 + \frac{x^2}{2} - 5x\Big|_5^7$

$= -\frac{25}{2} + 25 - (-8 + 20) + \frac{49}{2} - 35 - \left(\frac{25}{2} - 25\right)$

$= -\frac{25}{2} + 25 - 12 + \frac{49}{2} - 35 - \frac{25}{2} + 25 = -\frac{50}{2} + 3 + \frac{49}{2}$

$= -\frac{1}{2} + 3 = \boxed{\frac{5}{2}}$

*See me for a diagram. Compute the area bounded above the x-axis and subtract off the area bounded below the x-axis.

Turn in your own solutions.