

Worksheet 5, Tuesday, February 20th, 2018

Definition: the **Definite Integral** of a function f from $x = a$ to $x = b$ is given by

$$\begin{aligned}
 (**) \quad \int_a^b f(x) \, dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x \\
 &= \lim_{n \rightarrow \infty} [f(x_1) \Delta x + f(x_2) \Delta x + f(x_3) \Delta x + \dots + f(x_i) \Delta x + \dots + f(x_n) \Delta x]
 \end{aligned}$$

Note: The definite integral is a limit of a sum! Just think about this formula as

the limiting value of the sum of the areas of finitely many (n) approximating rectangles.

To compute definite integrals the *long (limit) way*, **follow these steps**:

Step 1: Given the integral $\int_a^b f(x) \, dx$, **pick off** or **identify** the **integrand** $f(x)$, and **limits of integration** a and b .

Step 2: Compute $\Delta x = \frac{b-a}{n}$. This width of each partitioned interval should be in terms of n .

Step 3: Compute $x_i = a + i \Delta x$. Leave the i as your counter. You have the left-most endpoint a from Step 1. You have width Δx from Step 2. This endpoint x_i should be in terms of i and n .

Step 4: Plug x_i and Δx into the formula (***) above. Here it is again:

$$(***) \quad \int_a^b f(x) \, dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x \quad \leftarrow \text{MEMORIZE!}$$

Step 5: Use the following formulas for sum of integers i and finish evaluating the limit in n .

$$\begin{aligned}
 \sum_{i=1}^n 1 &= n \\
 (*) \quad \sum_{i=1}^n i &= \frac{n(n+1)}{2} \\
 (**) \quad \sum_{i=1}^n i^2 &= \frac{n(n+1)(2n+1)}{6} \\
 (***) \quad \sum_{i=1}^n i^3 &= \left(\frac{n(n+1)}{2} \right)^2
 \end{aligned}$$

Note: your final answer for the definite integral should be a **number** after you finish the limit.

1. **Read** through the entire next problem. Make sure you understand the formula to start, as well as the formulas for Δx and x_i . Because it doesn't feel natural yet, just trust the formulas right now.

Evaluate $\int_0^6 x^2 dx$ using Riemann Sums.

Here $f(x) = x^2$, $a = 0$, $b = 6$, $\Delta x = \frac{b-a}{n} = \frac{6-0}{n} = \frac{6}{n}$ and $x_i = a+i\Delta x = 0+i\left(\frac{6}{n}\right) = \frac{6i}{n}$.

$$\begin{aligned}
 \int_0^6 x^2 dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x = \lim_{n \rightarrow \infty} \sum_{i=1}^n f\left(\frac{6i}{n}\right) \frac{6}{n} \\
 &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\left(\frac{6i}{n}\right)^2 \right) \frac{6}{n} \\
 &= \lim_{n \rightarrow \infty} \frac{6}{n} \sum_{i=1}^n \frac{36i^2}{n^2} \\
 &= \lim_{n \rightarrow \infty} \left(\frac{216}{n^3} \sum_{i=1}^n i^2 \right) \\
 &= \lim_{n \rightarrow \infty} \left(\frac{216}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \right) \text{ using } (**) \\
 &= \lim_{n \rightarrow \infty} \left(\frac{216}{6} \cdot \frac{n(n+1)(2n+1)}{n^3} \right) \\
 &= \lim_{n \rightarrow \infty} \left(\frac{216}{6} \cdot \frac{n(n+1)(2n+1)}{n \cdot n \cdot n} \right) \\
 &= \lim_{n \rightarrow \infty} \left(\frac{216}{6} \cdot \left(\frac{n}{n}\right) \cdot \left(\frac{n+1}{n}\right) \cdot \left(\frac{2n+1}{n}\right) \right) \\
 &= \lim_{n \rightarrow \infty} \left(\frac{216}{6} \cdot 1 \cdot \left(1 + \frac{1}{n}\right) \cdot \left(2 + \frac{1}{n}\right) \right) \\
 &= \frac{216}{6} \cdot 1 \cdot 2 = \frac{216}{3} = \boxed{72}
 \end{aligned}$$

2. Evaluate $\int_0^4 x - 1 dx$ using Riemann Sums.

3. Evaluate $\int_0^2 x^2 - 5x dx$ using Riemann Sums.

4. **Read** through the entire next problem. Make sure you understand the formula to start, as well as the formulas for Δx and x_i . Here the lower limit of integration a is **not** 0.

Evaluate $\int_1^4 6 - 3x \, dx$ using Riemann Sums.

Here $f(x) = 6 - 3x$, $a = 1$, $b = 4$, $\Delta x = \frac{b-a}{n} = \frac{4-1}{n} = \frac{3}{n}$

and $x_i = a + i\Delta x = 1 + i\left(\frac{3}{n}\right) = 1 + \frac{3i}{n}$.

$$\begin{aligned}\int_1^4 6 - 3x \, dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i)\Delta x = \lim_{n \rightarrow \infty} \sum_{i=1}^n f\left(1 + \frac{3i}{n}\right) \frac{3}{n} \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(6 - 3\left(1 + \frac{3i}{n}\right)\right) \frac{3}{n} \\ &= \lim_{n \rightarrow \infty} \left(\frac{3}{n} \sum_{i=1}^n \left(3 - \frac{9i}{n}\right)\right) \\ &= \lim_{n \rightarrow \infty} \left(\frac{3}{n} \left(\sum_{i=1}^n 3 - \sum_{i=1}^n \frac{9i}{n}\right)\right) \\ &= \lim_{n \rightarrow \infty} \left(\frac{9}{n} \sum_{i=1}^n 1 - \frac{27}{n^2} \sum_{i=1}^n i\right) \\ &= \lim_{n \rightarrow \infty} \left(\frac{9}{n}(n) - \frac{27}{n^2} \frac{n(n+1)}{2}\right) \text{ using } (*) \\ &= \lim_{n \rightarrow \infty} \left(9 - \frac{27}{2} \left(\frac{n}{n}\right) \left(\frac{n+1}{n}\right)\right) \\ &= \lim_{n \rightarrow \infty} \left(9 - \frac{27}{2}(1) \left(1 + \frac{1}{n}\right)\right) \\ &= 9 - \frac{27}{2} \\ &= \boxed{-\frac{9}{2}}\end{aligned}$$

Turn in your own solutions for Problems 2 and 3.