Math 106, Spring 2018

Answer Key

Worksheet 3, Tuesday, February 6th, 2018

Definition: Take F, f functions defined on an interval I and suppose that $F'(x) = f(x)$ on I. Then
• $F(x)$ is called **an** antiderivative of $f(x)$
• $F(x) + C$ is called the **most general** antiderivative of $f(x)$, where C is any constant.

We will use the notation $\displaystyle\int f(x)\ dx$ to denote the most general antiderivative.

For example: $\displaystyle\int x^7\ dx = \frac{x^8}{8} + C$ where $+C$ represent all possible constants.

Note that $\dfrac{x^8}{8} + 3$ is **an** antiderivative of x^7. So is $\dfrac{x^8}{8} + 2014$ as well as $\dfrac{x^8}{8} - 5$ and $\dfrac{x^8}{8} + \sqrt{3}$.

Hint: if you ever want to know whether you found the correct antiderivative, take the deriative of your answer and check that you return to the original function.

1. Write a *general power rule* for $\displaystyle\int x^n\ dx$ where n is any real number with $n \neq -1$.

   (We will learn the $n = -1$ case at the very end of this semester.)

   $$\int x^n\ dx = \boxed{\frac{x^{n+1}}{n+1} + C}$$

2. Compute $\displaystyle\int \sqrt{x}\ dx = \int x^{\frac{1}{2}}\ dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C = \boxed{\frac{2}{3}x^{\frac{3}{2}} + C}$

3. Compute $\displaystyle\int \frac{1}{x^9}\ dx = \int x^{-9}\ dx = \frac{x^{-8}}{-8} + C = \boxed{-\frac{1}{8x^8} + C}$

4. Compute $\displaystyle\int \frac{1}{\sqrt{x}}\ dx = \int x^{-\frac{1}{2}}\ dx = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C = \boxed{2x^{\frac{1}{2}} + C}$

5. Compute $\displaystyle\int \frac{1}{x^{\frac{3}{7}}}\ dx = \int x^{-\frac{3}{7}}\ dx = \frac{x^{\frac{4}{7}}}{\frac{4}{7}} + C = \boxed{\frac{7}{4}x^{\frac{4}{7}} + C}$

6. Compute $\displaystyle\int \cos x\ dx = \boxed{\sin x + C}$

7. Compute $\displaystyle\int \sin x\ dx = \boxed{-\cos x + C}$

8. Compute $\displaystyle\int \sec^2 x\ dx = \boxed{\tan x + C}$

9. Compute $\displaystyle\int \sec x \tan x\ dx = \boxed{\sec x + C}$

10. Find the general antiderivative of the following functions:

(a) $f(x) = x^3 + \dfrac{1}{\sqrt{x}} + 2$

$$\int x^3 + \frac{1}{\sqrt{x}} + 2\; dx = \int x^3 + x^{-\frac{1}{2}} + 2\; dx = \frac{x^4}{4} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + 2x + C = \boxed{\frac{x^4}{4} + 2x^{\frac{1}{2}} + 2x + C}$$

(b) $f(x) = x^3(1+x^2) = x^3 + x^5$ (simplify first, then antidifferentiate)

$$\int x^3(1+x^2)\; dx = \int x^3 + x^5\; dx = \boxed{\frac{x^4}{4} + \frac{x^6}{6} + C}$$

(c) $f(x) = \dfrac{x + \sqrt{x} + 7}{x^3} = \dfrac{x}{x^3} + \dfrac{\sqrt{x}}{x^3} + \dfrac{7}{x^3} = x^{-2} + x^{-\frac{5}{2}} + 7x^{-3}$

$$\int \frac{x+\sqrt{x}+7}{x^3}\; dx = \int x^{-2} + x^{-\frac{5}{2}} + 7x^{-3}\; dx = \frac{x^{-1}}{-1} + \frac{x^{-\frac{3}{2}}}{-\frac{3}{2}} + 7\left(\frac{x^{-2}}{-2}\right) + C$$

$$= \boxed{-x^{-1} - \frac{2}{3}x^{-\frac{3}{2}} - \frac{7}{2}x^{-2} + C} \overset{\text{or}}{=} \boxed{-\frac{1}{x} - \frac{2}{3x^{\frac{3}{2}}} - \left(\frac{7}{2x^2}\right) + C}$$

(d) $f(x) = x^2 + x(1+x)^2 = x^2 + x(1+2x+x^2) = x^2 + x + 2x^2 + x^3 = x + 3x^2 + x^3$

$$\int x^2 + x(1+x)^2\; dx = \int x + 3x^2 + x^3\; dx = \frac{x^2}{2} + 3\left(\frac{x^3}{3}\right) + \frac{x^4}{4} + C = \boxed{\frac{x^2}{2} + x^3 + \frac{x^4}{4} + C}$$

(e) $f(w) = \dfrac{w + \sqrt{w}}{\sqrt[4]{w^3}} = \dfrac{w + \sqrt{w}}{w^{\frac{3}{4}}} = \dfrac{w}{w^{\frac{3}{4}}} + \dfrac{\sqrt{w}}{w^{\frac{3}{4}}} = w^{\frac{1}{4}} + w^{\frac{1}{2}-\frac{3}{4}} = w^{\frac{1}{4}} + w^{-\frac{1}{4}}$

$$\int w^{\frac{1}{4}} + w^{-\frac{1}{4}}\; dw = \frac{w^{\frac{5}{4}}}{\frac{5}{4}} + \frac{w^{\frac{3}{4}}}{\frac{3}{4}} + C = \boxed{\frac{4}{5}w^{\frac{5}{4}} + \frac{4}{3}w^{\frac{3}{4}} + C}$$

(f) $f(x) = -3\cos x - \sec^2 x - 7\sec x \tan x$

$$\int -3\cos x - \sec^2 x - 7\sec x\tan x\; dx = \boxed{-3\sin x - \tan x - 7\sec x + C}$$

11. Is $\frac{1}{6}(x^2+1)^3$ an antiderivative of $x(x^2+1)^2$? Hint: use the definition of antiderivative...

We can check if it is an antiderivative by taking the Derivative of $\frac{1}{6}(x^2+1)^3$ and see if that answer equals $x(x^2+1)^2$.

Check: $\dfrac{d}{dx}\,\dfrac{1}{6}(x^2+1)^3 = \dfrac{3}{6}(x^2+1)^2(2x) = \dfrac{1}{2}(x^2+1)^2(2x) = x(x^2+1)^2$

$\boxed{\textbf{YES it is an antiderivative.}}$

12. Consider the curve $y = 2x + \sin x$. Explain why the tangent lines of this curve are never horizontal.

    $y' = 2 + \cos x$. For this equation to have horizontal tangent line, we need $y' = 2 + \cos x = 0$ which would require $\cos x = -2$. Since the range of the cosine function is $[-1, 1]$, the cosine graph never has output value less than -1. There is no solution to $\cos x = -2$

13. Find an antiderivative $F(x)$ of $2 + \sin x$ that satisfies $F\left(\frac{\pi}{2}\right) = 3$.

    First we antidifferentiate $F(x) = 2x - \cos x + C$

    Next, using the initial condition $F\left(\frac{\pi}{2}\right) = 3$ we have
    $F\left(\frac{\pi}{2}\right) = 2\left(\frac{\pi}{2}\right) - \cos\left(\frac{\pi}{2}\right) + C \overset{\text{set}}{=} 3$
    That is, $\pi - 0 + C = 3 \;\Rightarrow\; C = 3 - \pi$.

    Finally, plugging this specific constant of antidifferentiation into the general antiderivative above we see
    $F(x) = \boxed{2x - \cos x + 3 - \pi}$

14. Find the indicated functions:

    (a) $f(x)$ where $f'(x) = x^2 + 1$ and $f(1) = 3$.

    First find the general antiderivative
    $f(x) = \dfrac{x^3}{3} + x + C$.

    Next use the initial condition $f(1) = \dfrac{1}{3} + 1 + C \overset{\text{set}}{=} 3$.

    Solve for C. Here $C = 3 - \dfrac{4}{3} = \dfrac{5}{3}$.

    Finally, combining these pieces $f(x) = \boxed{\dfrac{x^3}{3} + x + \dfrac{5}{3}}$.

    (b) $g(t)$ where $g'(t) = t(2 + \sqrt{t})$ and $g(4) = 30$.

    Notice that $g'(t) = t(2 + \sqrt{t}) = 2t + t\sqrt{t} = 2t + t^{\frac{3}{2}}$

    First find the general antiderivative
    $g(t) = t^2 + \dfrac{t^{\frac{5}{2}}}{\frac{5}{2}} = t^2 + \dfrac{2}{5}t^{\frac{5}{2}} + C$

Next use the initial condition
$g(4) = 4^2 + \frac{2}{5}(4)^{\frac{5}{2}} + C = 16 + \frac{2}{5}(\sqrt{4})^5 + C = 16 + \frac{64}{5} + C \overset{\text{set}}{=} 30$

This implies $16 + \frac{64}{5} + C \overset{\text{set}}{=} 30$
$\frac{80}{5} + \frac{64}{5} + C = 30$
$\frac{144}{5} + C = \frac{150}{5}$

Solve for $C = \frac{6}{5}$. Here

Finally, combining these pieces $g(t) = \boxed{t^2 + \frac{2}{5}t^{\frac{5}{2}} + \frac{6}{5}}$.

(c) $h(t)$ where $h''(t) = \frac{1}{\sqrt{t}} + 3t^2$ and $h'(1) = 2, \quad h(1) = 0$.

First we will reverse the second derivative.

Rewrite
$h''(t) = \frac{1}{\sqrt{t}} + 3t^2 = t^{-\frac{1}{2}} + 3t^2$
$h'(t) = \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + t^3 + C_1 = 2t^{\frac{1}{2}} + t^3 + C_1$

Use the first bit of given information. $h'(1) = 2$.
$h'(1) = 2(1)^{\frac{1}{2}} + 1^3 + C_1 = 2 + 1 + C_1 \overset{\text{set}}{=} 2 \Rightarrow C_1 = -1$

Finally, $h'(t) = 2t^{\frac{1}{2}} + t^3 - 1$.

Next we will reverse the first derivative.
$h(t) = 2\frac{t^{\frac{3}{2}}}{\frac{3}{2}} + \frac{t^4}{4} - t + C_2 = \frac{4}{3}t^{\frac{3}{2}} + \frac{t^4}{4} - t + C_2$.

Use the second bit of given information. $h(1) = 0$.
$h(1) = \frac{4}{3}(1)^{\frac{3}{2}} + \frac{(1)^4}{4} - 1 + C_2 = \frac{4}{3} + \frac{1}{4} - 1 + C_2 \overset{\text{set}}{=} 0 \Rightarrow \frac{16}{12} + \frac{3}{12} - \frac{12}{12} + C_2 = 0 \Rightarrow C_2 = -\frac{7}{12}$

Finally, $h(t) = \boxed{\frac{4}{3}t^{\frac{3}{2}} + \frac{t^4}{4} - t - \frac{7}{12}}$

15. CHALLENGE: Can you use a *guess and check* approach to compute the function $f(x)$ where $f'(x) = \dfrac{\sec x \tan x}{\sqrt{\sec x + 8}}$ and $f(0) = 7$? Check your answer.

This derivative looks like it came from application of the Chain Rule. Because of the square root in the denominator, it looks like it came from the derivative of a square root in the numerator. Finally, since the numerator $\sec x \tan x$ is the derivative of the $\sec x + 8$, it does appear that the Chain Rule was applied.

Finally, we suspect that $f(x) = 2\sqrt{\sec x + 8} + C$. Check this by taking the derivative using the Chain Rule.

Now use the initial condition

$f(0) = 2\sqrt{(\sec 0) + 8} + C = 2\sqrt{9} + C = 6 + C \overset{\text{set}}{=} 7 \Rightarrow C = 1$

Finally, piecing these together $f(x) = \boxed{2\sqrt{\sec x + 8} + 1}$

Check your answer by taking the derivative, AND testing that $f(0) = 7$.

NOTE: we will have a formal way of computing this antiderivative soon...

Turn in your own solutions.