Math 106, Spring 2018

Answer Key
Worksheet 2, Tuesday, January 30th, 2018

1. Compute $f'(x)$ where $f(x) = \sec x \tan x$.

   $f'(x) = \sec x \sec^2 x + \tan x \sec x \tan x = \boxed{\sec^3 x + \sec x \tan^2 x}$ or $\boxed{\sec x(\sec^2 x + \tan^2 x)}$

2. Compute $f'(x)$ where $f(x) = \dfrac{\sin x + \cos x}{\sec x \tan x}$. (Watch the denominator) Do not simplify.

   Quotient Rule (using #1 above for the derivative of the denominator)

   $$f'(x) = \boxed{\frac{\sec x \tan x(\cos x - \sin x) - (\sin x + \cos x)(\sec^3 x + \sec x \tan^2 x)}{(\sec x \tan x)^2}}$$

3. Compute $f'(x)$ where $f(x) = \cos^4(x^3 - 5)$.

   $f'(x) = 4\cos^3(x^3 - 5)(-\sin(x^3 - 5))(3x^2) = \boxed{-12x^2 \cos^3(x^3 - 5)\sin(x^3 - 5)}$

4. Compute the derivative $\dfrac{dy}{dx}$ for the curve $y^2 + \cos x = xy$.

   Implicitly differentiate:

   $\dfrac{d}{dx}\left(y^2 + \cos x\right) = \dfrac{d}{dx}(xy)$

   $2y\dfrac{dy}{dx} - \sin x = x\dfrac{dy}{dx} + y(1)$

   Isolate $\dfrac{dy}{dx}$:

   $2y\dfrac{dy}{dx} - x\dfrac{dy}{dx} = y + \sin x$

   Factor:

   $(2y - x)\dfrac{dy}{dx} = y + \sin x$

   Solve: $\dfrac{dy}{dx} = \boxed{\dfrac{y + \sin x}{2y - x}}$

5. Compute the equation of the tangent line to the curve $\sin(x^2y)+6\tan x+1=y^3$ at the point $(0,1)$.

Compute $\dfrac{dy}{dx}$.

$$\frac{d}{dx}\left(\sin(x^2y)+6\tan x+1\right)=\frac{d}{dx}\left(y^3\right)$$

(***) $\cos(x^2y)\left(x^2\dfrac{dy}{dx}+y(2x)\right)+6\sec^2 x=3y^2\dfrac{dy}{dx}$

Distribute:

$$x^2\cos(x^2y)\frac{dy}{dx}+2xy\cos(x^2y)+6\sec^2 x=3y^2\frac{dy}{dx}$$

Algebra:

$$x^2\cos(x^2y)\frac{dy}{dx}-3y^2\frac{dy}{dx}=-2xy\cos(x^2y)-6\sec^2 x$$

Factor:

$$\left(x^2\cos(x^2y)-3y^2\right)\frac{dy}{dx}=-2xy\cos(x^2y)-6\sec^2 x$$

Solve:

$$\frac{dy}{dx}=\boxed{\frac{-2xy\cos(x^2y)-6\sec^2 x}{x^2\cos(x^2y)-3y^2}}$$

(b) Compute the equation of the tangent line to this curve at the point $(0,1)$.

Point=$(0,1)$

Slope at $(0,1)=\left.\dfrac{dy}{dx}\right|_{(0,1)}=\dfrac{0-6\sec^2 0}{0-3}=\dfrac{-6}{-3}=2$

****

OR you can compute the slope more simply by plugging the point $(0,1)$ directly into (***) above without solving explicitly for $\dfrac{dy}{dx}$.

(***) $\cos(x^2y)\left(x^2\dfrac{dy}{dx}+y(2x)\right)+6\sec^2 x=3y^2\dfrac{dy}{dx}$

(***) $\cos(0)\left(0\dfrac{dy}{dx}+0\right)+6\sec^2 0=3(1)^2\dfrac{dy}{dx}$

so $0+6=3\dfrac{dy}{dx}$

Solve $\dfrac{dy}{dx}=\dfrac{6}{3}=2$

****

Point-Slope Form: $y-1=2(x-0)$ or $\boxed{y=2x+1}$

6. Let $f(x) = \dfrac{1}{2\tan^2 x} + \cos^2 x + \sec(2x)$. Find $f'\left(\dfrac{\pi}{6}\right)$. **Simplify** your answer to a single real number.

First, $f(x) = \dfrac{1}{2}\tan^{-2} x + \cos^2 x + \sec(2x)$

Then, $f'(x) = \dfrac{1}{2}(-2)\tan^{-3} x \, (\sec^2 x) - 2\cos x \sin x + 2\sec(2x)\tan(2x)$

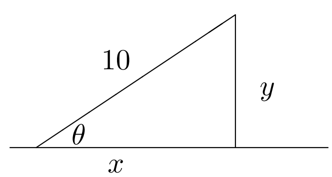
$= -1\tan^{-3} x \sec^2 x - 2\cos x \sin x + 2\sec(2x)\tan(2x)$

$$f'\left(\frac{\pi}{6}\right) = -\frac{1}{\tan^3\left(\frac{\pi}{6}\right)}\sec^2\left(\frac{\pi}{6}\right) - 2\cos\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{6}\right) + 2\sec\left(\frac{\pi}{3}\right)\tan\left(\frac{\pi}{3}\right)$$

$$= -\frac{1}{\left(\frac{1}{\sqrt{3}}\right)^3}\left(\frac{2}{\sqrt{3}}\right)^2 - 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) + 2(2)\left(\sqrt{3}\right) = -3\sqrt{3}\left(\frac{4}{3}\right) - \frac{\sqrt{3}}{2} + 4\sqrt{3} = \boxed{-\frac{\sqrt{3}}{2}}$$

7. The top of a ten foot ladder is sliding down a vertical wall at the rate of one foot every second. Consider the angle formed by the bottom of the ladder and the ground. How fast is this angle changing when the top of the ladder is three feet above the ground?

- Diagram



- Variables

Let x = distance between bottom of ladder and wall at time t
Let y = distance between top of ladder and ground at time t
Let θ = angle formed by the ground and base of ladder at time t

Given $\dfrac{dy}{dt} = -1\dfrac{\text{ft}}{\text{sec}}$,

Find $\dfrac{d\theta}{dt} =?$ when $y = 3$ ft

- Equation relating the variables:

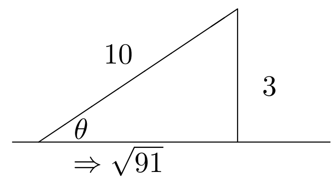
We have $\sin\theta = \dfrac{y}{10}$.

- Differentiate both sides w.r.t. time t.

$\dfrac{d}{dt}(\sin\theta) = \dfrac{d}{dt}\left(\dfrac{y}{10}\right) \implies \cos\theta\dfrac{d\theta}{dt} = \dfrac{1}{10}\dfrac{dy}{dt}$(Related Rates!)

- Extra Solvable Information

We're not given θ for this problem, but we can still compute $\cos\theta$ from trig. relations on the diagram's triangle with $\cos\theta = \frac{\text{adj}}{\text{hyp}}$. At the key moment, when $y = 3$, we can use the Pyth. Theorem to compute $x = \sqrt{(10)^2 - (3)^2} = \sqrt{91}$. Finally, $\cos\theta = \dfrac{\sqrt{91}}{10}$.



• Substitute Key Moment Information (now and not before now!):

$\cos\theta \frac{d\theta}{dt} = \frac{1}{10}\frac{dy}{dt}$

$\frac{\sqrt{91}}{10}\frac{d\theta}{dt} = \frac{1}{10}(-1)$

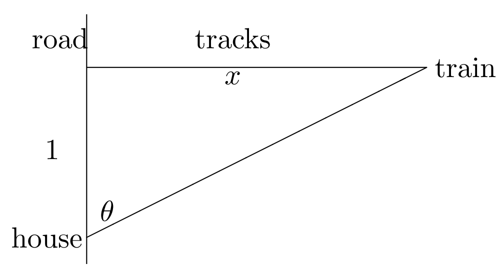
• Solve for the desired quantity:

$\frac{d\theta}{dt} = -\frac{1}{10} \cdot \frac{10}{\sqrt{91}} = -\frac{1}{\sqrt{91}}\frac{\text{rad}}{\text{sec}}$

• Answer the question that was asked: The angle is **decreasing** at a rate of $\frac{1}{\sqrt{91}}$ radians every second at this moment.

8. A train is travelling east on a straight track at 40 mph. The track is crossed by a road going north and south, and a house is on the road one mile south of the track. Draw the straight line connecting the house to the train. Consider the angle between the road and this line, as measured at the house. How fast is this angle changing when the train is 3 miles east of the road?

(a) • Diagram



The picture at arbitrary time t is:

• Variables

Let x = distance train has travelled horizontally(east) at time t

Given $\frac{dx}{dt} = 40\frac{\text{mi}}{\text{hr}}$

Find $\frac{d\theta}{dt} =?$ when $x = 3$ feet

• Equation relating the variables:

The trigonometry of the triangle yields $\tan\theta = \frac{x}{1}$

• Differentiate both sides w.r.t. time t.

$\frac{d}{dt}(\tan\theta) = \frac{d}{dt}(x) \implies \sec^2\theta\frac{d\theta}{dt} = \frac{dx}{dt}$

• Substitute Key Moment Information (now and not before now!):

At the key instant when $x = 3$, using the trigonometry of the triangle, we have a $1, 3, \sqrt{10}$ triangle snap shot. Then $\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{\sqrt{10}}{1}$

So, $\sec^2\theta \dfrac{d\theta}{dt} = \dfrac{dx}{dt}$ becomes

$(\sqrt{10})^2 \dfrac{d\theta}{dt} = 40$

• Solve for the desired quantity:

$\dfrac{d\theta}{dt} = \dfrac{40}{10} = 4\dfrac{\text{rad}}{\text{hr}}$

• Answer the question that was asked: The angle is **increasing** at 4 radians every hour at this moment.

9. CHALLENGE: Differentiate $\dfrac{d}{dx}\ \cos\left(\dfrac{\dfrac{6}{x^6}+\tan(3x)}{\dfrac{4}{x}+\sec x}\right)$

$= -\sin\left(\dfrac{\dfrac{6}{x^6}+\tan(3x)}{\dfrac{4}{x}+\sec x}\right)$ (Chain Rule continued below...)

$$\left[\frac{\left(\frac{4}{x}+\sec x\right)\left(-\frac{36}{x^7}+3\sec^2(3x)\right)-\left(\frac{6}{x^6}+\tan(3x)\right)\left(-\frac{4}{x^2}+\sec x\tan x\right)}{\left(\frac{4}{x}+\sec x\right)^2}\right]$$

Turn in your own solutions.