Answer Key

Math 106 Quiz #5 February 27, 2017

1. [10 points] Evaluate $\displaystyle\int_1^4 x^2 - 3x\ dx$ using the *limit definition of the definite integral.* Then draw a sketch of the bounded region and explain why the answer is negative. You may use the formulas at the bottom of the page.

Here $f(x) = x^2 - 3x$, $a = 1$, $b = 4$, $\Delta x = \dfrac{4-1}{n} = \dfrac{3}{n}$, $x_i = a + x_i = 1 + i\left(\dfrac{3}{n}\right) = 1 + \dfrac{3i}{n}$.

$$
\begin{aligned}
\int_1^4 x^2 - 3x\ dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(1 + \frac{3i}{n}\right)\left(\frac{3}{n}\right) \\
&= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left(1 + \frac{3i}{n}\right)^2 - 3\left(1 + \frac{3i}{n}\right) \\
&= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} 1 + \frac{6i}{n} + \frac{9i^2}{n^2} - 3 + \frac{9i}{n} \\
&= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \frac{9i^2}{n^2} - \frac{3i}{n} - 2 \\
&= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \frac{9i^2}{n^2} - \frac{3}{n} \sum_{i=1}^{n} \frac{3i}{n} - \frac{3}{n} \sum_{i=1}^{n} 2 \\
&= \lim_{n\to\infty} \frac{27}{n^3} \sum_{i=1}^{n} i^2 - \frac{9}{n^2} \sum_{i=1}^{n} i - \frac{6}{n} \sum_{i=1}^{n} 1 \\
&= \lim_{n\to\infty} \frac{27}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) - \frac{9}{n^2}\left(\frac{n(n+1)}{2}\right) - \frac{6}{n}(n) \\
&= \lim_{n\to\infty} \frac{27}{6}\left(\frac{n(n+1)(2n+1)}{n^3}\right) - \frac{9}{2}\left(\frac{n(n+1)}{n^2}\right) - 6 \\
&= \lim_{n\to\infty} \frac{27}{6}\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) - \frac{9}{2}\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right) - 6 \\
&= \lim_{n\to\infty} \frac{27}{6}(1)\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) - \frac{9}{2}(1)\left(1 + \frac{1}{n}\right) - 6 \\
&= \frac{54}{6} - \frac{9}{2} - 6 = 9 - \frac{9}{2} - 6 = 3 - \frac{9}{2} = \boxed{-\frac{3}{2}}
\end{aligned}
$$

See me for a sketch. The function $x^2 - 3x = x(x-3)$ has x-intercepts at $x = 0$ and $x = 3$. Note the answer is negative because more area is bounded below the x axis than bounded above the x-axis between $x = 1$ and $x = 4$.

2. [10 points] Evaluate $\int_0^3 2 - x \, dx$ using two different methods.

(a) First, use the *limit definition of the definite integral.*

Here $f(x) = 2 - x$, $a = 0$, $b = 3$, $\Delta x = \frac{3-0}{n} = \frac{3}{n}$, $x_i = a + x_i = 1 + i\left(\frac{3}{n}\right) = \frac{3i}{n}$.

$$\begin{aligned}
\int_0^3 2 - x \, dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\left(\frac{3}{n}\right) \\
&= \lim_{n\to\infty} \left(\frac{3}{n}\right) \sum_{i=1}^{n} 2 - \frac{3i}{n} \\
&= \lim_{n\to\infty} \left(\frac{3}{n}\right) \sum_{i=1}^{n} 2 - \left(\frac{3}{n}\right) \sum_{i=1}^{n} \frac{3i}{n} \\
&= \lim_{n\to\infty} \left(\frac{6}{n}\right) \sum_{i=1}^{n} 1 - \left(\frac{9}{n^2}\right) \sum_{i=1}^{n} i \\
&= \lim_{n\to\infty} \left(\frac{6}{n}\right)(n) - \left(\frac{9}{n^2}\right)\left(\frac{n(n+1)}{2}\right) \\
&= \lim_{n\to\infty} 6 - \left(\frac{9}{2}\right)\left(\frac{n(n+1)}{n^2}\right) \\
&= \lim_{n\to\infty} 6 - \left(\frac{9}{2}\right)\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right) \\
&= \lim_{n\to\infty} 6 - \left(\frac{9}{2}\right)(1)\left(1 + \frac{1}{n}\right) \\
& 6 - \frac{9}{2} = \boxed{\frac{3}{2}}
\end{aligned}$$

(b) Next, use Area Interpretation of the definite integral. Show your work.

See me for a sketch. The area bounded is two triangles, one above the x-axis and one below. The triangle bounded above has width 2 and height 2. so its area is 2, using $\frac{1}{2}b \cdot h$. The trainangle below has width 1 and height 1, so its area is $\frac{1}{2}$. Since the definite integral captures the net (signed) area, it computes the area bounded above the x-axis and subtracts the area bounded below the x-axis. Here that would be $2 - \frac{1}{2} = \frac{3}{2}$.