Answer Key

Math 106 | Quiz #10 | April 16, 2018

1. [5 points] Compute the derivative for $y = \ln\left(\dfrac{x^{\frac{3}{4}}\sqrt{\sec(4x)}}{(\cos x + \sin x)^6\; e^{\sqrt{3x+1}}}\right)$. Do not simplify.

First, we rewrite y using logarithmic algebraic properties.

$$y = \ln\left(x^{\frac{3}{4}}\sqrt{\sec(4x)}\right) - \ln\left((\cos x + \sin x)^6\; e^{\sqrt{3x+1}}\right)$$

$$= \ln\left(x^{\frac{3}{4}}\right) + \ln\left(\sqrt{\sec(4x)}\right) - \left(\ln\left((\cos x + \sin x)^6\right) + \ln\left(e^{\sqrt{3x+1}}\right)\right)$$

$$= \ln\left(x^{\frac{3}{4}}\right) + \ln\left(\sqrt{\sec(4x)}\right) - \ln\left((\cos x + \sin x)^6\right) - \ln\left(e^{\sqrt{3x+1}}\right)$$

$$= \frac{3}{4}\ln x + \frac{1}{2}\ln\left(\sec(4x)\right) - 6\ln(\cos x + \sin x) - \sqrt{3x+1}$$

Now differentiate, term-by-term, the simpler pieces.

$$y' = \boxed{\frac{3}{4}\left(\frac{1}{x}\right) + \frac{1}{2}\left(\frac{1}{\sec(4x)}\right)\sec(4x)\tan(4x)(4) - 6\left(\frac{1}{\cos x + \sin x}\right)(-\sin x + \cos x) - \frac{3}{2\sqrt{3x+1}}}$$

2. [5 points] Compute the derivative $\dfrac{dy}{dx}$ when $x^2e^y = x\ln y + \ln(e^9+1)$.

Implicitly differentiate both sides with respect to x.

$$\frac{d}{dx}\left(x^2e^y\right) = \frac{d}{dx}\left(x\ln y + \ln(e^9+1)\right)$$

$$x^2e^y\frac{dy}{dx} + e^y(2x) = x\left(\frac{1}{y}\right)\frac{dy}{dx} + \ln y(1) + 0$$

Isolate $\dfrac{dy}{dx}$

$$x^2e^y\frac{dy}{dx} - \left(\frac{x}{y}\right)\frac{dy}{dx} = \ln y - 2xe^y$$

Factor $\dfrac{dy}{dx}$

$$\left(x^2e^y - \frac{x}{y}\right)\frac{dy}{dx} = \ln y - 2xe^y$$

Solve

$$\frac{dy}{dx} = \boxed{\frac{\ln y - 2xe^y}{x^2e^y - \dfrac{x}{y}}}$$

3. [5 points] Find the function $f(x)$ that satsifies $f'(x) = \dfrac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}$ and $f(0) = 5$.

$$f(x) = \int f'(x)\,dx = \int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|e^{2x} + e^{-2x}| + C$$

Use the condition $f(0) = 5$ to solve for C.

$$f(0) = \frac{1}{2}\ln|e^0 + e^0| + C = \frac{1}{2}\ln 2 + C \overset{\text{set}}{=} 5 \Rightarrow C = 5 - \frac{\ln 2}{2}$$

Finally, $f(x) = \boxed{\frac{1}{2}\ln|e^{2x} + e^{-2x}| + 5 - \frac{\ln 2}{2}}$

$$\boxed{\begin{aligned} u &= e^{2x} + e^{-2x} \\ du &= 2e^{2x} - 2e^{-2x}dx \\ \frac{1}{2}du &= e^{2x} - e^{-2x}dx \end{aligned}}$$

4. [5 points] Compute $\displaystyle\int_0^{\ln 2} (e^x - e^{-2x})^2\,dx$. Simplify.

Use Algebra to simplify integrand.

$$\int_0^{\ln 2} (e^x - e^{-2x})^2\,dx = \int_0^{\ln 2} e^{2x} - 2e^{-x} + e^{-4x}\,dx = \frac{e^{2x}}{2} + 2e^{-x} - \frac{e^{-4x}}{4}\bigg|_0^{\ln 2}$$

$$= \left(\frac{e^{2\ln 2}}{2} + 2e^{-\ln 2} - \frac{e^{-4\ln 2}}{4}\right) - \left(\frac{e^0}{2} + 2e^0 - \frac{e^0}{4}\right)$$

$$= \left(\frac{e^{2\ln 2}}{2} + 2e^{-\ln 2} - \frac{e^{-4\ln 2}}{4}\right) - \left(\frac{1}{2} + 2 - \frac{1}{4}\right)$$

$$= \left(\frac{e^{\ln(2^2)}}{2} + 2e^{\ln(2^{-1}} - \frac{e^{\ln(2^{-4})}}{4}\right) - \left(\frac{1}{2} + 2 - \frac{1}{4}\right)$$

$$= \left(\frac{4}{2} + 2\left(\frac{1}{2}\right) - \frac{1}{64}\right) - \left(\frac{1}{2} + 2 - \frac{1}{4}\right)$$

$$= 2 + 1 - \frac{1}{64} - \frac{1}{2} - 2 + \frac{1}{4} = 1 - \frac{1}{64} - \frac{1}{4} = \frac{64}{64} - \frac{1}{64} - \frac{16}{64} = \boxed{\frac{47}{64}}$$

5. [5 points] Compute $\displaystyle\int_{e^5}^{e^{10}} \frac{1}{x\sqrt{(\ln x)-1}}\,dx$. Simplify.

$$\int_{e^5}^{e^{10}} \frac{1}{x\sqrt{(\ln x)-1}}\,dx = \int_4^9 \frac{1}{\sqrt{u}}\,du = 2\sqrt{u}\Big|_4^9 = 2\left(\sqrt{9}-\sqrt{4}\right) = 2(3-2) = \boxed{2}$$

$$\boxed{\begin{aligned} u &= (\ln x)-1 \\ du &= \frac{1}{x}dx \end{aligned}}$$

$$\boxed{\begin{aligned} x = e^5 &\Rightarrow u = (\ln e^5)-1 = 5-1 = 4 \\ x = e^{10} &\Rightarrow u = (\ln e^{10})-1 = 10-1 = 9 \end{aligned}}$$

6. [5 points] Compute $\displaystyle\int_0^{\frac{\pi}{6}} \tan x\;dx$. Simplify.

$$\int_0^{\frac{\pi}{6}} \tan x\;dx = \int_0^{\frac{\pi}{6}} \frac{\sin x}{\cos x}\,dx = -\int_1^{\frac{\sqrt{3}}{2}} \frac{1}{u}\,du = -\ln|u|\Big|_1^{\frac{\sqrt{3}}{2}} = -\left(\ln\left|\frac{\sqrt{3}}{2}\right| - \ln|1|\right) = \boxed{-\ln\left|\frac{\sqrt{3}}{2}\right|}$$

$$\boxed{\begin{aligned} u &= \cos x \\ du &= -\sin x dx \\ -du &= \sin x dx \end{aligned}}$$

$$\boxed{\begin{aligned} x = 0 &\Rightarrow u = \cos 0 = 1 \\ x = \tfrac{\pi}{6} &\Rightarrow u = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \end{aligned}}$$