Name: Answer Key

**Amherst College**

**DEPARTMENT OF MATHEMATICS**

**Math 106**

**Midterm Exam #3**

**April 23, 2014**

• This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, webpages, or other aids are permitted.

• Simplify numerical answers such as $\sin\left(\frac{\pi}{6}\right)$, $\ln^{e^3}$, $e^{2\ln 3}$ and $4^{\frac{3}{2}}$.

• Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

| Problem | Score | Possible Points |
|---|---|---|
| 1 | | 30 |
| 2 | | 10 |
| 3 | | 60 |
| Total | | 100 |

**1.** [30 Points] Compute each of the following derivatives.

(a) $y'$ where $y = \ln\left(\dfrac{(\sin^2 x)\ \sqrt{1+\sec\sqrt{x}}}{(5-x^7)^{\frac{3}{7}}\ e^{-\cos x}}\right)$ Do not simplify your final answer here.

Simplify using log Properties

$$y = \ln\left[\sin^2 x\sqrt{1+\sec\sqrt{x}}\right] - \ln\left[(5-x^7)^{3/7}e^{-\cos x}\right]$$

$$= \ln(\sin^2 x) + \ln\left[(1+\sec\sqrt{x})^{1/2}\right] - \ln\left[(5-x^7)^{3/7}\right] - \cancel{\ln e}^{-\cos x}$$

$$= 2\ln(\sin x) + \frac{1}{2}\ln(1+\sec\sqrt{x}) - \frac{3}{7}\ln(5-x^7) + \cos x$$

$$y' = \boxed{2\left(\frac{1}{\sin x}\right)\cdot\cos x + \frac{1}{2}\left(\frac{1}{1+\sec\sqrt{x}}\right)\sec\sqrt{x}\tan\sqrt{x}\left(\frac{1}{2\sqrt{x}}\right) - \frac{3}{7}\left(\frac{1}{5-x^7}\right)(-7x^6) - \sin x}$$

(b) $\dfrac{d}{dx}(\tan x)^{\sqrt{x}}$

Set $y = (\tan x)^{\sqrt{x}}$

$$\ln y = \ln\left[(\tan x)^{\sqrt{x}}\right] = \sqrt{x}\ln(\tan x)$$

Implicit Differentiation

$$\frac{1}{y}\frac{dy}{dx} = \sqrt{x}\left(\frac{1}{\tan x}\right)\sec^2 x + \ln(\tan x)\left(\frac{1}{2\sqrt{x}}\right)$$

Solve $\boxed{\dfrac{dy}{dx} = y\left[\dfrac{\sqrt{x}\sec^2 x}{\tan x} + \dfrac{\ln(\tan x)}{2\sqrt{x}}\right]}$

$(\tan x)^{\sqrt{x}}$

**1.** (Continued) Compute the following derivative.

(c) $\frac{dy}{dx}$ where $xe^{x+y} + \cos x = \ln(e+5) + y \ln y + x^2$ Implicitly Differentiate

$$\frac{d}{dx}\left[xe^{x+y}\right] = \frac{d}{dx}\left[\overbrace{\ln(e+5)}^{\text{constant}} + y\ln y + x^2\right]$$

$$xe^{x+y}\left[1+\frac{dy}{dx}\right] + e^{x+y} - \sin x = 0 + \cancel{y\left(\frac{1}{y}\right)}\frac{dy}{dx} + \ln y\left(\frac{dy}{dx}\right) + 2x$$

$$xe^{x+y} + xe^{x+y}\frac{dy}{dx} + e^{x+y} - \sin x = \frac{dy}{dx} + \ln y\,\frac{dy}{dx} + 2x$$

Isolate.

$$xe^{x+y}\frac{dy}{dx} - \frac{dy}{dx} - \ln y\,\frac{dy}{dx} = 2x - xe^{x+y} - e^{x+y} + \sin x$$

Factor

$$\left[xe^{x+y} - 1 - \ln y\right]\frac{dy}{dx} = 2x - xe^{x+y} - e^{x+y}$$

Solve

$$\frac{dy}{dx} = \boxed{\frac{2x - xe^{x+y} - e^{x+y} + \sin x}{xe^{x+y} - 1 - \ln y}}$$

OR $$\frac{dy}{dx} = \boxed{\frac{xe^{x+y} + e^{x+y} - 2x - \sin x}{\ln y + 1 - xe^{x+y}}}$$

**2.** [10 Points] Find the equation of the tangent line to the curve

$$y = \ln(1+\cos x) + \cos(\ln(1+x)) - e^{\sin x} + \frac{e}{1+\ln(x+1)} + e^{x+1} \cdot \cos(e^x - 1) - \ln 2$$

at the point where $x = 0$.

$$y(0) = \ln(1+\cos 0) + \cos(\ln(1+0)) - e^{\sin 0} + \frac{e}{1+\ln(0+1)} + e^{0+1} \cdot \cos(e^0 - 1) - \ln 2$$

$$= \ln 2 + \cos 0 - e^0 + e + e \cdot \cos 0 - \ln 2$$

$$= \cancel{\ln 2} + \cancel{1} - \cancel{1} + e + e - \cancel{\ln 2}$$

$$= \boxed{2e}$$

$$y' = \frac{1}{1+\cos x}(-\sin x) - \sin(\ln(1+x))\left(\frac{1}{1+x}\right) - e^{\sin x}(\cos x) - \frac{e}{(1+\ln(x+1))^2}\left(\frac{1}{x+1}\right)$$

$$+ e^{x+1}(-\sin(e^x - 1))e^x + \cos(e^x - 1)\, e^{x+1} - 0$$

$$y'(0) = \frac{-\sin 0}{1+\cos 0} - \frac{\sin(\ln(1+0))}{1+0} - e^{\sin 0}(\cos 0) - \frac{e}{(1+\ln(0+1))^2}\left(\frac{1}{0+1}\right)$$

$$- e^{0+1}\sin(e^0 - 1)e^0 + \cos(e^0 - 1)\, e^{0+1}$$

$$= 0 - 0 - 1 - \cancel{e} - 0 + \cancel{e}$$

$$= \boxed{-1}$$

Point Slope Form

Point $= (0, 2e)$

Slope $= -1$

$$y - 2e = -1(x - 0) \Rightarrow \boxed{y = -x + 2e}$$

**3.** [60 Points] Evaluate each of the following integrals. Simplify.

(a) $\int_0^{\ln 2}\left(e^x+\frac{1}{e^x}\right)\left(1+\frac{1}{e^{2x}}\right)\,dx = \int_0^{\ln 2} e^x + \frac{1}{e^x} + \frac{1}{e^x} + \frac{1}{e^{3x}}\,dx$

$$= \int_0^{\ln 2} e^x + 2e^{-x} + e^{-3x}\,dx = e^x - 2e^{-x} - \frac{1}{3}e^{-3x}\Big|_0^{\ln 2}$$

$$= \left(\cancel{e}^{\cancel{\ln} 2} - 2e^{-\ln 2} - \frac{1}{3}e^{-3\ln 2}\right) - \left(\underbrace{\overset{1}{e^0} - 2\overset{1}{e^0}}_{-1} - \frac{1}{3}\overset{1}{e^0}\right)$$

$$= 2 - 2\cancel{e}^{\cancel{\ln}(2^{-1})} - \frac{1}{3}\underbrace{e^{\ln(2^{-3})}}_{1/8} + 1 + \frac{1}{3}$$

$$= 2 - \cancel{1} - \frac{1}{24} + \cancel{1} + 1/3 = \frac{48}{24} - \frac{1}{24} + \frac{8}{24} = \boxed{\frac{55}{24}}$$

(b) $\int_1^{\sqrt{3}} \frac{w}{4-w^2}\,dw$

$\boxed{\begin{array}{l} u = 4-w^2 \\ du = -2w\,dw \\ -\frac{1}{2}du = w\,dw \end{array}}$

$\boxed{\begin{array}{l} w=1 \Rightarrow u = 4-1=3 \\ w=\sqrt{3} \Rightarrow u = 4-3=1 \end{array}}$

$$= -\frac{1}{2}\int_3^1 \frac{1}{u}\,du = -\frac{1}{2}\ln|u|\Big|_3^1$$

watch order

$$= -1/2\left[\cancel{\ln 1}^{\,0} - \ln 3\right]$$

$$= \boxed{\frac{1}{2}\ln 3}$$

OR $= \boxed{\ln\sqrt{3}}$

$4^{3/2} = (\sqrt{4})^3 = 2^3 = 8$

**3.** (Continued) Evaluate each of the following integrals. Simplify.

(c) $\displaystyle\int_1^{e^3} \frac{\sqrt{4-\ln x}}{x}\,dx = -\int_4^1 \sqrt{u} = -\frac{2}{3}u^{3/2}\Big|_4^1$

$= -\frac{2}{3}\left[\cancel{1^{3/2}}^{1} - \cancel{4^{3/2}}^{8}\right]$

$= -\frac{2}{3}\left[1-8\right]$

$= -\frac{2}{3}(-7) = \boxed{\frac{14}{3}}$

$u = 4 - \ln x$

$du = -\frac{1}{x}\,dx$

$-du = \frac{1}{x}\,dx$

$x = 1 \Rightarrow u = 4 - \cancel{\ln 1}^{0} = 4$

$x = e^3 \Rightarrow u = 4 - \ln(e^3) = 4 - 3 = 1$

(d) $\displaystyle\int \frac{1}{xe^{\ln x}}\,dx = \int \frac{1}{x\cdot x}\,dx = \int \frac{1}{x^2}\,dx = \int x^{-2}\,dx = \boxed{\frac{-1}{x} + C}$

OR

$u = \ln x$

$du = \frac{1}{x}\,dx$

$= \displaystyle\int \frac{1}{e^u}\,du = \int e^{-u}\,du = -e^{-u} + C$

$= -\frac{1}{e^{\ln x}} + C$

$= \boxed{-\frac{1}{x} + C}$ Match!

**3.** (Continued) Evaluate each of the following integrals. Simplify.

(e) $\int \frac{\left(x^{\frac{3}{4}}-1\right)\left(x^3-x^{\frac{5}{4}}\right)}{x^3}\,dx = \int \frac{x^{15/4}-x^2-x^3+x^{5/4}}{x^3}\,dx$

$= \int \frac{x^{15/4}}{x^3} - \frac{x^2}{x^3} - \cancel{\frac{x^3}{x^3}} + \frac{x^{5/4}}{x^3}\,dx = \int x^{3/4} - \frac{1}{x} - 1 + x^{-7/4}\,dx$

$= \boxed{\frac{4}{7}x^{7/4} - \ln|x| - x - \frac{4}{3}x^{-3/4} + C}$

(f) $\int_1^4 \frac{1}{\sqrt{x}\,e^{1+\sqrt{x}}}\,dx = 2\int_2^3 \frac{1}{e^u}\,du = 2\int_2^3 e^{-u}\,du = -2e^{-u}\Big|_2^3$

$u = 1+\sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$2du = \frac{1}{\sqrt{x}}\,dx$

$x=1 \Rightarrow u = 1+\sqrt{1} = 2$

$x=4 \Rightarrow u = 1+\sqrt{4} = 3$

$= \frac{-2}{e^u}\Big|_2^3 = \frac{-2}{e^3} - \left(\frac{-2}{e^2}\right)$

$= \boxed{\frac{2}{e^2} - \frac{2}{e^3}}$

************************************************************************

# OPTIONAL BONUS

Do not attempt this unless you are completely done with the rest of the exam.

************************************************************************

**OPTIONAL BONUS #1** Compute $\int e^{(e^{x+e^x}+x+e^x)}(1+e^x)\,dx$

$$= \int e^{\overbrace{e^{x+e^x}}^{u}} \cdot \underbrace{e^{x+e^x}(1+e^x)\,dx}_{du}$$

$$\boxed{\begin{aligned} u &= e^{x+e^x} \\ du &= e^{x+e^x}(1+e^x)\,dx \end{aligned}}$$

$$= \int e^u\,du$$

$$= e^u + C$$

$$= \boxed{e^{e^{x+e^x}} + C}$$