Math 106 Review Packet for Exam #3 [also HW #20]

1. $y = x^{\sqrt{x}}$ Compute y'

$\ln y = \ln\left[x^{\sqrt{x}}\right]$

$\ln y = \sqrt{x}\ln x$

$\frac{d}{dx}\left[\ln y\right] = \frac{d}{dx}\left[\sqrt{x}\ln x\right]$

$\frac{1}{y}\frac{dy}{dx} = \cancel{\sqrt{x}} \cdot \frac{1}{\cancel{x}_{\sqrt{x}}} + \ln x \cdot \frac{1}{2\sqrt{x}}$

Solve

$$\frac{dy}{dx} = y\left[\frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}}\right]$$

$$= \boxed{x^{\sqrt{x}}\left[\frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}}\right]}$$

2. $f(x) = (\tan x)^x$ Compute $f'(x)$. Call it $f(x)$ or y.

$\ln\left[f(x)\right] = \ln\left[(\tan x)^x\right]$

$\ln\left[f(x)\right] = x\ln(\tan x)$

$\frac{d}{dx}\left[\ln(f(x))\right] = \frac{d}{dx}\left[x\ln(\tan x)\right]$

$\frac{1}{f(x)} \cdot f'(x) = x \cdot \frac{1}{\tan x} \cdot \sec^2 x + \ln(\tan x)(1)$

Solve $f'(x) = f(x)\left[\frac{x\sec^2 x}{\tan x} + \ln(\tan x)\right]$

$$= \boxed{(\tan x)^x\left[\frac{x\sec^2 x}{\tan x} + \ln(\tan x)\right]}$$

3. $f(x) = 5^x$

$y = 5^x$

$\ln y = \ln\left[5^x\right]$

constant

$\ln y = x \ln 5 = (\ln 5)\, x$

$\frac{d}{dx}\left(\ln y\right] = \frac{d}{dx}\left[(\ln 5) x\right]$

$\frac{1}{y}\frac{dy}{dx} = \ln 5$

Solve $\frac{dy}{dx} = y \cdot \ln 5 = \boxed{5^x \cdot \ln 5}$

4. $f(x) = \tan\left(\ln(1+x^2)\right) + \ln(1+\sec^2 x) + \frac{5}{\ln(1+x^2)}$ $\quad 5\left[\ln(1+x^2)\right]^{-1}$

$f'(x) = \sec^2\left[\ln(1+x^2)\right] \cdot \frac{1}{1+x^2}(2x) + \frac{1}{1+\sec^2 x} \cdot 2\sec x \cdot (\sec x \tan x) \ldots$

(continued) $\ldots - 5\left[\ln(1+x^2)\right]^{-2} \cdot \frac{1}{1+x^2}(2x)$

5. $f(x) = \dfrac{x^4}{e^x}$

Quotient Rule

$$f'(x) = \frac{e^x(4x^3) - x^4e^x}{(e^x)^2} = \frac{\cancel{e^x}(4x^3 - x^4)}{e^{\cancel{2x}}} = \frac{4x^3 - x^4}{e^x}$$

$$f''(x) = \frac{e^x[12x^2 - 4x^3] - (4x^3 - x^4)e^x}{(e^x)^2}$$

$$= \frac{\cancel{e^x}(12x^2 - 4x^3 - 4x^3 + x^4)}{e^{\cancel{2x}}} = \boxed{\frac{x^4 - 8x^3 + 12x^2}{e^x}}$$

OR Product + Chain Rule

$$f(x) = x^4e^{-x}$$

$$f'(x) = x^4(-e^{-x}) + e^{-x}(4x^3) = e^{-x}(4x^3 - x^4)$$

$$f''(x) = e^{-x}(12x^2 - 4x^3) + (-e^{-x})(4x^3 - x^4)$$

$$= e^{-x}\left[12x^2 - 4x^3 - 4x^3 + x^4\right]$$

$$= \boxed{\frac{x^4 - 8x^3 + 12x^2}{e^x}}$$

6. $g(x) = \displaystyle\int_x^{2012} \sqrt{\ln t} + \ln\sqrt{t}\ dt = -\int_{2012}^{x} \sqrt{\ln t} + \ln\sqrt{t}\ dt$

$g'(x) = -\left(\sqrt{\ln x} + \ln\sqrt{x}\right)$ FTC Part 1 "Derivative undoes integral"

$$g''(x) = -\left[\frac{1}{2\sqrt{\ln x}}\left(\frac{1}{x}\right) + \frac{1}{\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\right)\right] = \boxed{-\left[\frac{1}{2x\sqrt{\ln x}} + \frac{1}{2x}\right]}$$

7. $$\frac{d}{dx} \ln\left[\frac{(x^2+1)^{4/7}\, e^{\tan x}}{\sqrt{1+\sqrt{x}}}\right]$$

$$= \frac{d}{dx}\left[\ln\left[(x^2+1)^{4/7} e^{\tan x}\right] - \ln\sqrt{1+\sqrt{x}}\right]$$

Algebraic Properties of Log

$$= \frac{d}{dx}\left[\ln\left[(x^2+1)^{4/7}\right] + \cancel{\ln}\left[\cancel{e}^{\tan x}\right] - \ln\left[(1+\sqrt{x})^{1/2}\right]\right]$$

$$= \frac{d}{dx}\left[\frac{4}{7}\ln(x^2+1) + \tan x - \frac{1}{2}\ln(1+\sqrt{x})\right]$$

$$= \frac{4}{7}\cdot\left(\frac{1}{x^2+1}\right)(2x) + \sec^2 x - \frac{1}{2}\cdot\left(\frac{1}{1+\sqrt{x}}\right)\cdot\frac{1}{2\sqrt{x}}$$

$$= \boxed{\frac{8x}{7(x^2+1)} + \sec^2 x - \frac{1}{4\sqrt{x}(1+\sqrt{x})}}$$

8. $\sin y + e^x = \sec x + \cancel{\cos(e^9)} - e^{xy}$

constant

Implicit Differentiation $\frac{d}{dx}$ Both Sides

$$\cos y \frac{dy}{dx} + e^x = \sec x \tan x - e^{xy}\left[x\frac{dy}{dx} + y(1)\right]$$

$$= \sec x \tan x - xe^{xy}\frac{dy}{dx} - ye^{xy}$$

Isolate $\quad \cos y \frac{dy}{dx} + xe^{xy}\frac{dy}{dx} = \sec x \tan x - ye^{xy} - e^x$

Factor $\quad \left(\cos y + xe^{xy}\right)\frac{dy}{dx} = \sec x \tan x - ye^{xy} - e^x$

Solve $\quad \frac{dy}{dx} = \boxed{\dfrac{\sec x \tan x - ye^{xy} - e^x}{\cos y + xe^{xy}}}$

9. $$\int \frac{(3-\sqrt{x})(1+2\sqrt{x})}{x^2}\,dx = \int \frac{3+5\sqrt{x}-2x}{x^2}\,dx = \int \frac{3}{x^2} + \frac{5\sqrt{x}}{x^2} - \frac{2x}{x^2}\,dx$$

Algebra First to Prepare for Power + $\ln x$ Rules

$$= \int 3x^{-2} + 5x^{-3/2} - \frac{2}{x}\,dx$$

$$= \frac{3x^{-1}}{(-1)} + 5\frac{x^{-1/2}}{\left(-\frac{1}{2}\right)} - 2\ln|x| + C$$

$$= \boxed{-\frac{3}{x} - \frac{10}{\sqrt{x}} - 2\ln|x| + C}$$

10. $$\int_e^{e^3} \frac{4}{x(\ln x)^2}\,dx = 4\int_1^3 \frac{1}{u^2}\,du = \frac{4u^{-1}}{(-1)}\Big|_1^3 = \frac{-4}{u}\Big|_1^3 = -4\left(\frac{1}{3}-\frac{1}{1}\right)$$

$$= \frac{-4}{3}+4 = \boxed{\frac{8}{3}}$$

$u=\ln x$
$du = \frac{1}{x}\,dx$

$x=e \Rightarrow u=\ln e = 1$
$x=e^3 \Rightarrow u=\ln(e^3)=3$

11. $$\int_{\pi/2}^{\pi} \frac{\sin x}{e+\cos x}\,dx = -\int_e^{e-1} \frac{1}{u}\,du = -\ln|u|\Big|_e^{e-1} = -\left[\ln|e-1| - \cancel{\ln e}^{\,1}\right]$$

$$= \boxed{-\ln(e-1)+1}$$

$u = e+\cos x$
$du = -\sin x\,dx$
$-du = \sin x\,dx$

$x=\pi/2 \Rightarrow u = e+\cancel{\cos(\pi/2)}^{\,0} = e$
$x=\pi \Rightarrow u = e+\cancel{\cos\pi}^{\,-1} = e-1$

12. $$\int_{\ln 3}^{\ln 8} \frac{e^x}{\sqrt{1+e^x}}\,dx = \int_4^9 \frac{1}{\sqrt{u}}\,du = 2\sqrt{u}\Big|_4^9 = 2\left[\cancel{\sqrt{9}}^{\,3} - \cancel{\sqrt{4}}^{\,2}\right] = 2\left[\cancel{3-2}^{\,1}\right] = \boxed{2}$$

$u = 1+e^x$
$du = e^x\,dx$

$x=\ln 3 \Rightarrow u = 1+\cancel{e^{\ln 3}}^{\,3} = 1+3 = 4$
$x=\ln 8 \Rightarrow u = 1+\cancel{e^{\ln 8}}_{\,8} = 1+8 = 9$

13. $$\int \frac{\sec(e^{-x})\tan(e^{-x})}{e^x}\,dx = -\int \sec u\tan u\,du = -\sec u + C = \boxed{-\sec(e^{-x})+C}$$

$u = e^{-x}$
$du = -e^{-x}\,dx$
$-du = \frac{1}{e^x}\,dx$

14. $\int \tan(3x)\,dx = \int \frac{\sin(3x)}{\cos(3x)}\,dx = -\frac{1}{3}\int \frac{1}{u}\,du = -\frac{1}{3}\ln|u| + C = \boxed{-\frac{1}{3}\ln|\cos(3x)| + }$

$$\boxed{\begin{array}{l} u=\cos(3x) \\ du=-3\sin(3x)\,dx \\ -\frac{1}{3}du=\sin(3x)\,dx \end{array}}$$

15 $y=\cos(\ln(x+1))+\ln(\cos x)+e^{\sin x}+\sin(e^x-1)$

$y(0)=\cos(\ln 1)+\ln(\cos 0)+e^{\sin 0}+\sin(e^0-1)$

$1 + 0 + 1 + 0 = 2$ y-coordinate

Point: $(0,2)$

Derivative:

$$y' = -\sin(\ln(x+1))\cdot\left(\frac{1}{x+1}\right)+\frac{1}{\cos x}\cdot(-\sin x)+e^{\sin x}\cdot(\cos x)+\cos(e^x-1)(e^x)$$

$$y'(0) = -\sin(\ln 1)\cdot\frac{1}{1}+\frac{1}{\cos 0}(-\sin 0)+e^{\sin 0}(\cos 0)+\cos(e^0-1)(e^0)$$

$0 + 0 + 1\cdot 1 \quad 1\cdot 1$

$=0+0+1+1=2$ Specific Slope

Point-Slope Form

$y-2=2(x-0)$

$\boxed{y=2x+2}$ Equation of Tangent at $x=0$.

Point $(0, y(0)) = (0, 2)$

because $y(0) = \cos(\ln(0+1)) + \ln(\cos 0) + e^{\sin 0} + \sin(e^0 - 1) = \cos 0 + \ln 1 + e^0 + \sin 0$

$= 1 + 0 + 1 + 0 = 2$

Point-Slope Form

$y - 2 = 2(x - 0)$

$\boxed{y = 2x + 2}$

Curve Sketching

8. [20 Points] Let $f(x) = \dfrac{x^4}{e^x} = x^4 e^{-x}$.

For this function, discuss domain, vertical and horizontal asymptote(s), interval(s) of increase or decrease, local extreme value(s), concavity, and inflection point(s). Then use this information to present a detailed and labelled sketch of the curve.

Take my word that $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = +\infty$.

Also take my word that $f'(x) = \dfrac{x^3(4-x)}{e^x}$ and $f''(x) = \dfrac{x^2(x-2)(x-6)}{e^x}$.

- Domain$= \mathbb{R}$. Note that $f(x) = \dfrac{x^4}{e^x}$ and e^x is never zero in the denominator.
- It has no vertical asymptotes.
- There is a horizontal asymptote for this f at $y = 0$ because $\lim_{x\to\infty} f(x) = 0$.
- First Derivative Information

We use the given derivative $f'(x) = \dfrac{x^3(4-x)}{e^x}$ to find critical numbers. The critical points occur where f' is undefined (never here) or zero. The latter happens when $x = 0$ or $x = 4$.

Using sign testing/analysis for f',

f'	$\ominus$		$\oplus$		$\ominus$
		0		4	
f	$\searrow$	local min	$\nearrow$	local max	$\searrow$

So f is increasing on the interval $(0, 4)$; and f is decreasing on $(-\infty, 0)$ and $(4, \infty)$. Moreover, f has a local max at $x = 4$ with $f(4) = 256e^{-4}$, and a local min at $x = 0$ with $f(0) = 0$.

- Second Derivative Information

Setting $f'' = 0$ we solve for our possible inflection points $x = 0, x = 2$, or $x = 6$.

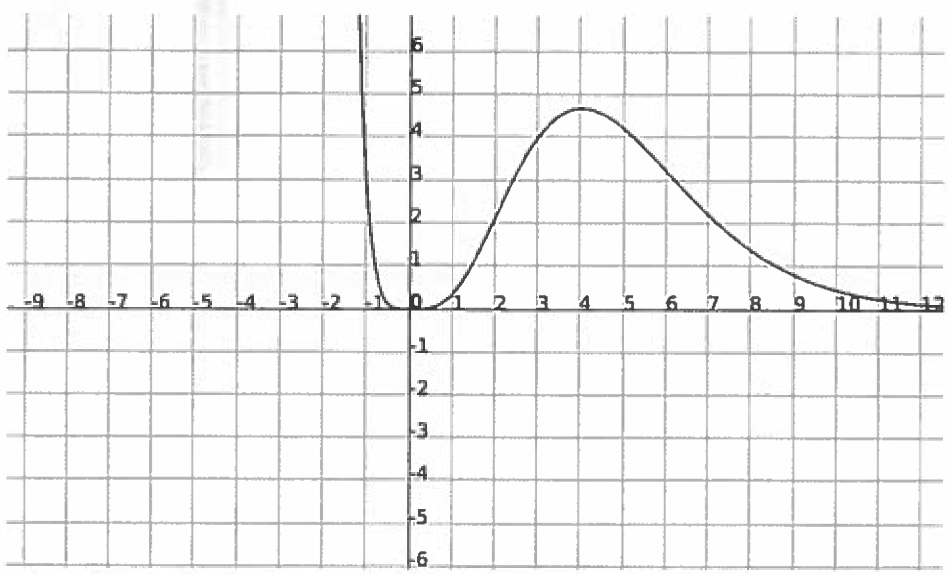
Using sign testing/analysis for f'',

		0		2		6	
f''	$\oplus$		$\oplus$		$\ominus$		$\oplus$
f	$\cup$	0	$\cup$	2	$\cap$	6	$\cup$
				infl. point		infl. point	

So f is concave down on the interval $(2,6)$ and concave up on $(-\infty,2)$ and $(6,\infty)$, with inflection points at $x=2$ and $x=6$.

- Piece the first and second derivative information together

		0		2		4		6	
f	$\searrow$		$\nearrow$		$\nearrow$		$\searrow$		$\searrow$
f	$\cup$		$\cup$		$\cap$		$\cap$		$\cup$
f		local min		infl. pt.		local max		infl. pt.	



~~9. [15 Points] A conical tank, 14 feet across the entire top and 12 feet deep, is leaking water.~~