Math 106 Practice Exam #3 Spring 2014

1. $f(x) = \ln(x + \ln x)$

$$f'(x) = \frac{1}{x+\ln x}\left(1+\frac{1}{x}\right) = \frac{1+\frac{1}{x}}{x+\ln x}$$ Chain Rule

$$f''(x) = \frac{(x+\ln x)(-1/x^2) - (1+\frac{1}{x})(1+\frac{1}{x})}{(x+\ln x)^2}$$ Quotient Rule + Algebra

$$= \frac{\frac{-1}{x} - \frac{\ln x}{x^2} - 1 - \frac{2}{x} - \frac{1}{x^2}}{(x+\ln x)^2}$$ Common Denominator

$$= \frac{\frac{-x}{x^2} - \frac{\ln x}{x^2} - \frac{x^2}{x^2} - \frac{2x}{x^2} - \frac{1}{x^2}}{(x+\ln x)^2}$$

$$= \frac{\frac{-x^2-3x-\ln x-1}{x^2}}{(x+\ln x)^2} = \boxed{\frac{-[x^2+3x+\ln x+1]}{x^2(x+\ln x)^2}}$$ ✓

2. $g(x) = e^x + \frac{1}{e^x} + x^e + \frac{1}{x^e} + \frac{x}{e} + \frac{e}{x} + ex + \frac{1}{ex} + e^{e^x+x} + e^{x-e} + (x-e)^e$

$$= e^x + e^{-x} + x^e + x^{-e} + \frac{1}{e}x + ex^{-1} + ex + \frac{1}{e}x^{-1} + e^{e^x+x} + e^{x-e} + (x-e)^e$$

$$\boxed{g'(x) = e^x - e^{-x} + ex^{e-1} - ex^{-e-1} + \frac{1}{e} - ex^{-2} + e - \frac{1}{e}x^{-2} + e^{e^x+x}(e^x+1) + e^{x-e}(1) + e(x-e)^{e-1}(1)}$$

3. $ye^{x+y} + \sec^2 x = \ln(3x) + \sec y$

Implicitly Differentiation

$$ye^{x+y}\left(1+\frac{dy}{dx}\right) + e^{x+y}\frac{dy}{dx} + 2\sec x\cdot(\sec x\tan x) = \frac{1}{\cancel{3}x}(\cancel{3}) + \sec y\tan y\frac{dy}{dx}$$

$$ye^{x+y} + ye^{x+y}\frac{dy}{dx} + e^{x+y}\frac{dy}{dx} + 2\sec^2 x\tan x = \frac{1}{x} + \sec y\tan y\frac{dy}{dx}$$

$$ye^{x+y}\frac{dy}{dx} + e^{x+y}\frac{dy}{dx} - \sec y\tan y\frac{dy}{dx} = \frac{1}{x} - 2\sec^2 x\tan x - ye^{x+y}$$

$$\left[ye^{x+y} + e^{x+y} - \sec y\tan y\right]\frac{dy}{dx} = \frac{1}{x} - 2\sec^2 x\tan x - ye^{x+y}$$

Solve $\dfrac{dy}{dx} = \boxed{\dfrac{\frac{1}{x} - 2\sec^2 x\tan x - ye^{x+y}}{ye^{x+y} + e^{x+y} - \sec y\tan y}}$

4. $y = x^{\sin x}$

$$\ln y = \ln\left[x^{\sin x}\right]$$

$$\ln y = \sin x \ln x$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left[\sin x\cdot\ln x\right]$$

$$\frac{1}{y}\frac{dy}{dx} = \sin x\left(\frac{1}{x}\right) + \ln x\cdot\cos x$$

Solve $\dfrac{dy}{dx} = y\left[\dfrac{\sin x}{x} + \ln x\cdot\cos x\right]$

replace

$$\frac{dy}{dx} = \boxed{x^{\sin x}\left[\frac{\sin x}{x} + \ln x\cdot\cos x\right]}$$

5. $\dfrac{d}{dx} \ln\left[\dfrac{e^{-\sin x}\sqrt{1+\sec\sqrt{x}}}{(5-x^7)^{-2/3}}\right]$

$= \dfrac{d}{dx} \ln\left(e^{-\sin x}\sqrt{1+\sec\sqrt{x}}\right) - \ln\left[(5-x^7)^{-2/3}\right]$

$= \dfrac{d}{dx} \cancel{\ln e}^{-\sin x} + \ln\sqrt{1+\sec\sqrt{x}} - \ln\left[(5-x^7)^{-2/3}\right]$

$= \dfrac{d}{dx} -\sin x + \dfrac{1}{2}\ln(1+\sec\sqrt{x}) + \dfrac{2}{3}\ln(5-x^7)$

$= -\cos x + \dfrac{1}{2}\left(\dfrac{1}{1+\sec\sqrt{x}}\right)\sec\sqrt{x}\tan\sqrt{x}\left(\dfrac{1}{2\sqrt{x}}\right) + \dfrac{2}{3}\left(\dfrac{1}{5-x^7}\right)(-7x^6)$

$= \boxed{-\cos x + \dfrac{\sec\sqrt{x}\tan\sqrt{x}}{4\sqrt{x}(1+\sec\sqrt{x})} - \dfrac{14x^6}{3(5-x^7)}}$

6. $f(x) = \dfrac{e^{-x}\sin(e^x)}{e^x\cos(e^{-x})}$

← Note: Could also use Logarithmic Differentiation

$f'(x) = -2 + \dfrac{e^x\cos(e^x)}{\sin(e^x)} - \dfrac{e^{-x}\sin(e^{-x})}{\cos(e^{-x})}$

$f'(x) = \dfrac{e^x\cos(e^{-x})\left[e^{-x}\cos(e^x)e^x + \sin(e^x)(-e^{-x})\right] - e^{-x}\sin(e^x)\left[e^x(-\sin(e^{-x}))(-e^{-x}) + \cos(e^{-x})e^x\right]}{\left[e^x\cos(e^{-x})\right]^2}$

7. $g(x) = \sqrt{x}\ln x + \ln\sqrt{x} + \sqrt{x\ln x} + \dfrac{1}{x\sqrt{\ln x}}$

$(x\sqrt{\ln x})^{-1}$

$$\boxed{g'(x) = \sqrt{x}\left(\frac{1}{x}\right) + \ln x\left(\frac{1}{2\sqrt{x}}\right) + \frac{1}{\sqrt{x}}\cdot\left(\frac{1}{2\sqrt{x}}\right) + \frac{1}{2\sqrt{x\ln x}}\left[x\left(\frac{1}{x}\right) + \ln x(1)\right] - (x\sqrt{\ln x})^{-2}\left[x\cdot\frac{1}{2\sqrt{\ln x}}\left(\frac{1}{x}\right) + \sqrt{\ln x}\,(1)\right]}$$

8. $\displaystyle\int \frac{(x^{1/3}+1)(1-x^{2/3})}{x^2}\,dx = \int \frac{x^{1/3} - x + 1 - x^{2/3}}{x^2}\,dx$

Algebra First

$$= \int \frac{x^{1/3}}{x^2} - \frac{x}{x^2} + \frac{1}{x^2} - \frac{x^{2/3}}{x^2}\,dx$$

$$= \int x^{-5/3} - \frac{1}{x} + x^{-2} - x^{-4/3}\,dx$$

$$= \frac{x^{-2/3}}{\left(-\frac{2}{3}\right)} - \ln|x| + \frac{x^{-1}}{(-1)} - \frac{x^{-1/3}}{\left(-\frac{1}{3}\right)} + C$$

$$= \boxed{\frac{-3}{2}x^{-2/3} - \ln|x| - \frac{1}{x} + 3x^{-1/3} + C}$$

9. $\displaystyle\int_{e^4}^{e^9} \frac{\sqrt{\ln x}}{3x}\,dx = \frac{1}{3}\int_4^9 \sqrt{u}\,du = \frac{1}{3}\left.\frac{u^{3/2}}{\left(\frac{3}{2}\right)}\right|_4^9 = \left.\frac{2}{9}u^{3/2}\right|_4^9$

$\boxed{\begin{array}{l} u = \ln x \\ du = \frac{1}{x}dx \end{array}}$ $\boxed{\begin{array}{l} x=e^4 \Rightarrow u = \cancel{\ln} \cancel{e}^4 = 4 \\ x = e^9 \Rightarrow u = \cancel{\ln} \cancel{e}^9 = 9 \end{array}}$ $= \frac{2}{9}\left[9^{3/2} - 4^{3/2}\right]$

$$= \frac{2}{9}\left[(\underbrace{\sqrt{9}}_{3})^3 - (\underbrace{\sqrt{4}}_{2})^3\right]$$

$$= \frac{2}{9}\left[27 - 8\right] = \boxed{\frac{38}{9}}$$

10. $\displaystyle\int_0^{\ln 2} \left(e^x + \frac{1}{e^{2x}}\right)^2 dx = \int_0^{\ln 2} e^{2x} + 2\frac{e^x}{e^{2x}} + \frac{1}{e^{4x}}\,dx = \int_0^{\ln 2} e^{2x} + 2e^{-x} + e^{-4x}\,dx$

$$= \left.\frac{1}{2}e^{2x} - 2e^{-x} - \frac{1}{4}e^{-4x}\right|_0^{\ln 2} = \left(\frac{1}{2}e^{2\ln 2} - \frac{2}{\underbrace{e^{\ln 2}}_{2}} - \frac{1}{4e^{4\ln 2}}\right) - \left(\underbrace{\frac{1}{2}e^{0}}_{\frac{1}{2}} - \underbrace{2e^{0}}_{-2} - \underbrace{\frac{1}{4}e^{0}}_{-\frac{1}{4}}\right) - \left(-\tfrac{7}{4}\right)$$

$$= \frac{1}{2}e^{\ln(2^2)} - 1 - \frac{1}{4e^{\ln(2^4)}} + \frac{7}{4} = \underbrace{2-1}_{1} - \frac{1}{64} + \frac{7}{4} = \frac{64}{64} - \frac{1}{64} + \frac{112}{64} = \boxed{\frac{175}{64}}$$

11. $\displaystyle\int e^x + \frac{1}{e^x} + x^e + \frac{1}{x^e} + \frac{x}{e} + \frac{e}{x} + ex + \frac{1}{ex} + e^{e^x+x} + e^{x-e} + (x-e)^e \, dx$

Rewrite a few pieces to prepare for antidifferentiation rule. See below for last 3 pieces.

$$= \int e^x + e^{-x} + x^e + x^{-e} + \frac{1}{e}x + e\left(\frac{1}{x}\right) + ex + \frac{1}{e}\left(\frac{1}{x}\right) + \underbrace{e^{e^x}e^x}_{(*)} + \underbrace{e^{x-e}}_{(**)} + \underbrace{(x-e)^e}_{(***)} \, dx$$

$$\boxed{= e^x - e^{-x} + \frac{x^{e+1}}{e+1} + \frac{x^{-e+1}}{-e+1} + \frac{1}{e}\left(\frac{x^2}{2}\right) + e\ln|x| + \frac{ex^2}{2} + \frac{1}{e}\ln|x| + e^{e^x} + e^{x-e} + \frac{(x-e)^{e+1}}{e+1} + C}$$

(*) $\displaystyle\int e^{e^x} e^x \, dx = \int e^u \, du = e^u + C = e^{e^x} + C$

$\boxed{\begin{array}{l} u = e^x \\ du = e^x dx \end{array}}$

(**) $\displaystyle\int e^{x-e} \, dx = \int e^u \, du = e^u + C = e^{x-e} + C$

$\boxed{\begin{array}{l} u = x-e \\ du = dx \end{array}}$

(***) $\displaystyle\int (x-e)^e \, dx = \int u^e \, du = \frac{u^{e+1}}{e+1} + C = \frac{(x-e)^{e+1}}{e+1} + C$

$\boxed{\begin{array}{l} u = x-e \\ du = dx \end{array}}$

12. $$\int_{\pi/18}^{\pi/9} \tan(3x)\,dx = \int_{\pi/18}^{\pi/9} \frac{\sin(3x)}{\cos(3x)}\,dx = -\frac{1}{3}\int_{\sqrt{3}/2}^{1/2} \frac{1}{u}\,du = -\frac{1}{3}\ln|u| \Big|_{\sqrt{3}/2}^{1/2}$$

$$\boxed{\begin{array}{l} u=\cos(3x) \\ du=-3\sin(3x)dx \\ -\frac{1}{3}du=\sin(3x)dx \end{array}}$$

$$= -\frac{1}{3}\left[\ln\left(\frac{1}{2}\right) - \ln\left(\frac{\sqrt{3}}{2}\right)\right]$$

$$= -\frac{1}{3}\left[\ln\left[\frac{\cancel{1}/\cancel{2}}{\sqrt{3}/\cancel{2}}\right]\right] = \boxed{-\frac{1}{3}\ln\left[\frac{1}{\sqrt{3}}\right]} \text{ OR } \boxed{\frac{1}{3}\ln\sqrt{3}}$$

$$\boxed{\begin{array}{l} x=\pi/18 \Rightarrow u=\cos\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2} \\ x=\pi/9 \Rightarrow u=\cos\left(\pi/3\right)=\frac{1}{2} \end{array}}$$

13. $$\int \frac{1}{e^{3x}(1+e^{-3x})^{2/9}}\,dx = -\frac{1}{3}\int \frac{1}{u^{2/9}}\,du = -\frac{1}{3}\,\frac{u^{7/9}}{(7/9)} + C = \boxed{\frac{9}{21}\left(1+e^{-3x}\right)^{7/9} + C}$$

$u^{-2/9}$

$$\boxed{\begin{array}{l} u=1+e^{-3x} \\ du=-3e^{-3x}dx \\ -\frac{1}{3}du=\frac{1}{e^{3x}}dx \end{array}}$$

14. $$\int \frac{e^x}{(1+e^x)^2}\,dx = \int \frac{1}{u^2}\,du = \frac{u^{-1}}{(-1)} + C = -\frac{1}{u} + C = \boxed{-\frac{1}{1+e^x} + C}$$

$$\boxed{\begin{array}{l} u=1+e^x \\ du=e^x dx \end{array}}$$

15. $$\int \frac{(1+e^x)^2}{e^x}\,dx = \int \frac{1+2e^x+e^{2x}}{e^x}\,dx = \int \frac{1}{e^x} + \frac{2\cancel{e^x}}{\cancel{e^x}} + \frac{e^{2x}}{e^x}\,dx = \int e^{-x}+2+e^x\,dx$$

$$= \boxed{-e^{-x} + 2x + e^x + C}$$

Note: Make sure you understand why algebra doesn't help for #14 and why u-substitution doesn't work for #15.

16. $$\int_0^1 \frac{x}{1+x^2}\,dx = \frac{1}{2}\int_1^2 \frac{1}{u}\,du = \frac{1}{2}\ln|u|\Big|_1^2 = \frac{1}{2}\left[\ln 2 - \cancel{\ln 1}^{\,0}\right] = \boxed{\frac{1}{2}\ln 2} = \boxed{\ln\sqrt{2}}$$

$$\boxed{\begin{aligned} u &= 1+x^2 \\ du &= 2x\,dx \\ \tfrac{1}{2}du &= x\,dx \end{aligned}} \quad \boxed{\begin{aligned} x=0 &\Rightarrow u=1 \\ x=1 &\Rightarrow u=2 \end{aligned}}$$

17. $$\int e^{x^2+\ln x+1}\,dx \underset{\text{Algebra}}{=} \int e^{x^2}\cdot e^{\ln x}\cdot e^{1}\,dx = \int e^{x^2}\cdot x\cdot e\,dx = \overset{\text{constant}}{e}\int x e^{x^2}\,dx$$

$$\boxed{\begin{aligned} u &= x^2 \\ du &= 2x\,dx \\ \tfrac{1}{2}du &= x\,dx \end{aligned}}$$

$$= \frac{e}{2}\int e^u\,du = \frac{e}{2}\left(e^u\right) + C$$

$$= \boxed{\frac{e}{2}\left(e^{x^2}\right) + C}$$

OR $$= \frac{e^{1+x^2}}{2} + C$$

Recall: Exponential Algebra

$$e^{a+b} = e^a \cdot e^b$$

18. $$f(x) = \frac{e^{3x}}{1+e^x} \qquad f(\ln 2) = \frac{e^{3\ln 2}}{1+e^{\ln 2}} = \frac{e^{\ln(2^3)}}{1+2} = \frac{8}{3} \quad \text{y-coordinate}$$

$$f'(x) = \frac{(1+e^x)e^{3x}(3) - e^{3x}(e^x)}{(1+e^x)^2}$$

$$f'(\ln 2) = \frac{(1+e^{\ln 2})e^{3\ln 2}(3) - e^{3\ln 2}e^{\ln 2}}{(1+e^{\ln 2})^2} = \frac{(1+2)(8)(3) - 8(2)}{3^2} = \frac{72-16}{9} = \overset{\text{slope}}{\frac{56}{9}}$$

Point-Slope Form

$$y - \tfrac{8}{3} = \tfrac{56}{9}(x - \ln 2) \qquad y = \boxed{\frac{56}{9}x - \frac{56}{9}(\ln 2) + \frac{8}{3}}$$

19. $f(x) = e^{\frac{-x^2}{2}}$ (Note: symmetry $\rightsquigarrow$ even function $f(x)=f(-x)$)

- Domain: $\mathbb{R}$

- V.A.: None
- H.A.: $y=0$ $\quad \lim_{x\to\pm\infty} e^{-x^2/2} = \lim_{x\to\pm\infty} \frac{1}{\underbrace{e^{x^2/2}}_{\infty}} = 0$

- First Derivative

$$f'(x) = e^{-x^2/2}\left(\frac{-\cancel{2}x}{\cancel{2}}\right) = -x\underbrace{e^{-x^2/2}}_{\oplus} \overset{set}{=} 0 \Rightarrow x=0 \text{ critical number}$$

f': $\oplus$ (test $x=-1$) | 0 | $\ominus$ (test $x=1$); $\nearrow$ local max $\searrow$

f Increasing $(-\infty, 0)$
Decreasing $(0, \infty)$
Local Max $(0, f(0)) = (0, 1)$

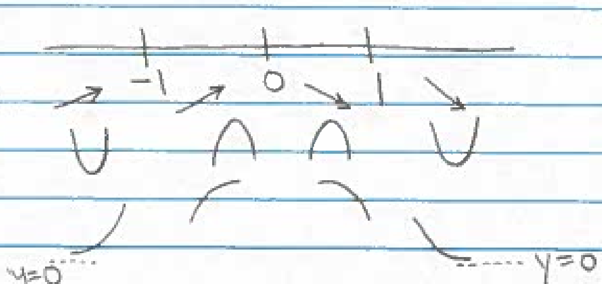
- Second Derivative

$$f''(x) = -xe^{-x^2/2}\left(\frac{-2x}{2}\right) + e^{\frac{-x^2}{2}}(-1)$$
$$= x^2e^{-x^2/2} - e^{-x^2/2}$$
$$= (x^2-1)\underbrace{e^{-x^2/2}}_{\oplus} \overset{set}{=} 0 \Rightarrow x^2=1 \Rightarrow x=\pm 1$$

	-10	-1	0	1	10
f''	$\oplus$		$\ominus$		$\oplus$
f	$\cup$	IP	$\cap$	IP	$\cup$

f Concave Up $(-\infty, -1) \cup (1, \infty)$
Concave Down $(-1, 1)$
I.P. $(\pm 1, e^{-1/2}) = (\pm 1$

- Piece Together



- Sketch