Name: Answer Key

Amherst College

DEPARTMENT OF MATHEMATICS

Math 106

Midterm Exam #2

March 26, 2018

- This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, webpages, or other aids are permitted.

- Simplify numerical answers such as $\sin\left(\frac{\pi}{6}\right)$ and $4^{\frac{3}{2}}$.

- Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

Problem	Score	Possible Points
1		18
2		4
3		32
4		32
5		4
6		10
Total		100

1. [18 Points] Compute $\int_{-1}^{2} 2 - 3x - x^2 \, dx$ using two different methods:

$+$ $\frown$ $-\;-$

(a) Fundamental Theorem of Calculus

$$\int_{-1}^{2} 2-3x-x^2dx = 2x - \frac{3x^2}{2} - \frac{x^3}{3}\Big|_{-1}^{2} = 4 - \frac{3(2)^2}{2} - \frac{2^3}{3} - \left(-2 - \frac{3(-1)^{\overset{1}{2}}}{2} - \frac{(-1)^{\overset{-1}{3}}}{3}\right)$$

$$= 4 - 6 - \frac{8}{3} - \left(-2 - 3/2 + \frac{1}{3}\right) = -\cancel{2} - 8/3 + 2 + \frac{\cancel{3}}{2} - \frac{1}{3} = -\frac{9}{3} + \frac{3}{2} = -3 + \frac{3}{2} = -\frac{6}{2} + \frac{3}{2} = \boxed{\frac{-3}{2}}$$

(b) Limit Definition of the Definite Integral.

$$\int_{-1}^{2} 2-3x-x^2dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(-1+\frac{3i}{n}\right)\left(\frac{3}{n}\right)$$

$1 - \frac{6i}{n} + \frac{9i^2}{n^2}$

$a = -1$, $b = 2$ $\Big\rangle$ $\Delta x = \frac{b-a}{n} = \frac{2-(-1)}{n} = \frac{3}{n}$

$x_i = a + i\Delta x = -1 + \frac{3i}{n}$

$$= \lim_{n\to\infty} \frac{3}{n}\sum_{i=1}^{n}\left[2 - 3\left(-1+\frac{3i}{n}\right) - \left(-1+\frac{3i}{n}\right)^2\right]$$

$$= \lim_{n\to\infty} \frac{3}{n}\sum_{i=1}^{n} 2 + 3 - \frac{9i}{n} - 1 + \frac{6i}{n} - \frac{9i^2}{n^2}$$

$$= \lim_{n\to\infty} \frac{3}{n}\sum_{i=1}^{n} 4 - \frac{3i}{n} - \frac{9i^2}{n^2} = \lim_{n\to\infty} \frac{3}{n}\sum_{i=1}^{n} 4 - \frac{3}{n}\sum_{i=1}^{n}\frac{3i}{n} - \frac{3}{n}\sum_{i=1}^{n}\frac{9i^2}{n^2}$$

$$= \lim_{n\to\infty} \frac{12}{n}\cancel{\sum_{i=1}^{n} 1}^{\,n} - \frac{9}{n^2}\sum_{i=1}^{n} i - \frac{27}{n^3}\sum_{i=1}^{n} i^2$$

$$= \lim_{n\to\infty} \frac{12}{\cancel{n}}(\cancel{n}) - \frac{9}{n^2}\left[\frac{n(n+1)}{2}\right] - \frac{27}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty} 12 - \frac{9}{2}\left(\frac{\cancel{n}}{\cancel{n}}\right)\left(\frac{n+1}{n}\right) - \frac{27}{6}\left(\frac{\cancel{n}}{\cancel{n}}\right)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)$$

$$= \lim_{n\to\infty} 12 - \frac{9}{2}\left(1 + \cancel{\tfrac{1}{n}}^{0}\right) - \frac{27}{6}\left(1 + \cancel{\tfrac{1}{n}}^{0}\right)\left(2 + \cancel{\tfrac{1}{n}}^{0}\right)$$

$$= 12 - 9/2 - \frac{\cancel{27}^{\,9}}{3} = 3 - 9/2 = \frac{6}{2} - \frac{9}{2} = \boxed{\frac{-3}{2}}$$

MATCH!

2. [4 Points] Compute $g'(x)$ where $g(x) = \int_x^3 \frac{\sec^2 t}{\sqrt{t^2 + 9\sin t}}\, dt$.

$$g'(x) = \frac{d}{dx} \int_x^3 \frac{\sec^2 t}{\sqrt{t^2+9\sin t}}\, dt$$

$$= -\frac{d}{dx} \int_3^x \frac{\sec^2 t}{\sqrt{t^2+9\sin t}}\, dt$$

$$= \boxed{-\frac{\sec^2 x}{\sqrt{x^2+9\sin x}}}$$

3. [32 Points] Evaluate each of the following integrals. Simplify.

(a) $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} 4\sin x \cdot \cos^3 x \, dx$ $= -4 \int_{\frac{\sqrt{3}}{2}}^{\frac{\sqrt{2}}{2}} u^3 du = -\frac{\cancel{4} u^4}{\cancel{4}} \Big|_{\frac{\sqrt{3}}{2}}^{\frac{\sqrt{2}}{2}} = -u^4 \Big|_{\frac{\sqrt{3}}{2}}^{\frac{\sqrt{2}}{2}}$

$u = \cos x$

$du = -\sin x \, dx$

$-du = \sin x \, dx$

$= -\left(\frac{\sqrt{2}}{2}\right)^4 \overset{--}{+} \left(\frac{\sqrt{3}}{2}\right)^4 = -\frac{4}{16} + \frac{9}{16} = \boxed{\frac{5}{16}}$

$x = \pi/6 \Rightarrow u = \cos(\pi/6) = \frac{\sqrt{3}}{2}$

$x = \pi/4 \Rightarrow u = \cos(\pi/4) = \frac{\sqrt{2}}{2}$

$(\sqrt{2})^4 = [(\sqrt{2})^2]^2 = 2^2 = 4$

$(\sqrt{3})^4 = [(\sqrt{3})^2]^2 = 3^2 = 9$

(b) $\int \frac{1}{\sqrt{x}\,\sqrt{2+\sqrt{x}}} \, dx$ $= 2\int \frac{1}{\sqrt{u}} du = 2\int u^{-1/2} du = \frac{2u^{1/2}}{1/2} + C$

$u = 2 + \sqrt{x}$

$du = \frac{1}{2\sqrt{x}} dx$

$2du = \frac{1}{\sqrt{x}} dx$

$= 4\sqrt{u} + C$

$= \boxed{4\sqrt{2+\sqrt{x}} + C}$

3. (Continued) Evaluate each of the following integrals. Simplify.

(c) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sec^2 x}{\tan^3 x}\,dx = \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{u^3}\,du = \int_{1/\sqrt{3}}^{\sqrt{3}} u^{-3}\,du = \frac{u^{-2}}{-2}\Big|_{1/\sqrt{3}}^{\sqrt{3}}$

$$\boxed{\begin{array}{l} u=\tan x \\ du=\sec^2 x\,dx \end{array}}$$

$$\boxed{\begin{array}{l} x=\pi/6 \Rightarrow u=\tan \pi/6 = \frac{1}{\sqrt{3}} \\ x=\pi/3 \Rightarrow u=\tan \pi/3=\sqrt{3} \end{array}}$$

$$= -\frac{1}{2u^2}\Big|_{1/\sqrt{3}}^{\sqrt{3}} = -\frac{1}{2}\left[\frac{1}{(\sqrt{3})^2} - \frac{1}{(1/\sqrt{3})^2}\right]$$

$$= -\frac{1}{2}\left[\frac{1}{3} - \underset{9/3}{3}\right] = -\frac{1}{2}\left[\frac{-8}{3}\right] = \boxed{\frac{4}{3}}$$

(d) $\int \frac{\sec\left(9+\frac{1}{x^2}\right)\tan\left(9+\frac{1}{x^2}\right)}{x^3}\,dx = -\frac{1}{2}\int \sec u \tan u\,du$

$$\boxed{\begin{array}{l} u = 9 + 1/x^2 = 9 + x^{-2} \\ du = -2x^{-3}dx \\ -\frac{1}{2}\,du = \frac{1}{x^3}\,dx \end{array}}$$

$$= -\frac{1}{2}\sec u + C$$

$$= \boxed{-\frac{1}{2}\sec\left(9+1/x^2\right) + C}$$

4. [32 Points] Evaluate each of the following integrals. Simplify.

(a) $\int_1^4 \frac{1-x}{\sqrt{x}}\,dx = \int_1^4 \frac{1}{\sqrt{x}} - \frac{x}{\sqrt{x}}\,dx = \int_1^4 x^{-1/2} - x^{1/2}\,dx$

SPLIT

$$= \frac{x^{1/2}}{1/2} - \frac{x^{3/2}}{3/2}\Big|_1^4 = 2\sqrt{x} - \frac{2}{3}x^{3/2}\Big|_1^4$$

$$= 2\overset{2}{\cancel{\sqrt{4}}} - \frac{2}{3}\overset{8}{\cancel{(4)^{3/2}}} - \left(2 - \frac{2}{3}\right) = 4 - \frac{16}{3} - 2 + \frac{2}{3} = 2 - \frac{14}{3}$$

$$= \frac{6}{3} - \frac{14}{3} = \boxed{\frac{-8}{3}}$$

$$4^{3/2} = (\underset{2}{\sqrt{4}})^3 = 2^3 = 8$$

(b) $\int_{-2}^{-1} \left(x - \frac{1}{x}\right)^2 dx = \int_{-2}^{-1} x^2 - 2 + \overset{x^{-2}}{\frac{1}{x^2}}\,dx = \frac{x^3}{3} - 2x + \frac{x^{-1}}{-1}\Big|_{-2}^{-1}$

FOIL:

$$\left(x - \frac{1}{x}\right)^2 = \left(x - \frac{1}{x}\right)\left(x - \frac{1}{x}\right)$$

$$= x^2 - 2 + \frac{1}{x^2}$$

$$= \frac{x^3}{3} - 2x - \frac{1}{x}\Big|_{-2}^{-1}$$

$$= \frac{(-1)^3}{3} - 2(-1) - \frac{1}{(-1)} - \left[\overset{-\frac{8}{3}}{\frac{(-2)^3}{3}} - \overset{+4}{2(-2)} - \overset{+\frac{1}{2}}{\frac{1}{(-2)}}\right]$$

$$= -\frac{1}{3} + 2 + 1 + \frac{8}{3} - 4 - \frac{1}{2}$$

$$= -1 + \frac{7}{3} - \frac{1}{2}$$

$$= -\frac{6}{6} + \frac{14}{6} - \frac{3}{6} = \boxed{\frac{5}{6}}$$

4. (Continued) Evaluate each of the following integrals. Simplify.

(c) $\int \sqrt{x} \cos(x\sqrt{x})\ dx$ $= \int \sqrt{x} \cos\left(x^{3/2}\right) dx$

$$\boxed{\begin{array}{l} u = x^{3/2} \\ du = \frac{3}{2} x^{1/2} dx \\ \frac{2}{3} du = x^{1/2} dx \end{array}}$$

$$= \frac{2}{3} \int \cos u \, du$$

$$= +\frac{2}{3} \sin u + C$$

$$= \boxed{\frac{2}{3} \sin\left(x^{3/2}\right) + C}$$

(d) $\int_{-3}^{-2} x(x+2)^7\ dx$ $= \int_{-1}^{0} (u-2)\,u^7\,du = \int_{-1}^{0} u^8 - 2u^7\,du$

Invert:

$$\boxed{\begin{array}{l} u = x+2 \Rightarrow x = u-2 \\ du = dx \end{array}}$$

$$\boxed{\begin{array}{l} x = -3 \Rightarrow u = -3+2 = -1 \\ x = -2 \Rightarrow u = -2+2 = 0 \end{array}}$$

$$= \frac{u^9}{9} - \frac{2u^8}{8} \Big|_{-1}^{0} = \frac{u^9}{9} - \frac{u^8}{4} \Big|_{-1}^{0}$$

$$= 0 - 0 - \left(\frac{\overset{-1}{(-1)^9}}{9} - \frac{\overset{1}{\cancel{(-1)^8}}}{4}\right)$$

$$= +\frac{1}{9} + 1/4$$

$$= \frac{4}{36} + \frac{9}{36} = \boxed{\frac{13}{36}}$$

5. [4 Points] Compute $f(x)$ where $f'(x) = \sin\left(\frac{x}{2}\right)$ and $f\left(-\frac{\pi}{3}\right) = 4\sqrt{3}$

$$f(x) = \int f'(x)\,dx$$

$$= \int \sin\left(\frac{x}{2}\right)dx = 2\int \sin u\,du = -2\cos u + C = -2\cos\left(\frac{x}{2}\right) + C$$

$u = \frac{x}{2}$

$du = \frac{1}{2}dx$

$2du = dx$

Test Point:

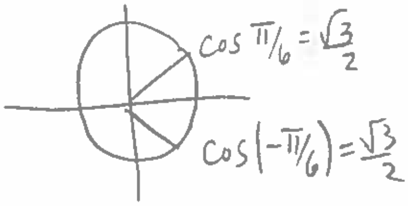
$$f\left(-\pi/3\right) = -2\cos\left(\frac{-\pi/3}{2}\right) + C$$

$$= -2\underbrace{\cos\left(-\pi/6\right)}_{\frac{\sqrt{3}}{2}} + C \overset{\text{set}}{=} 4\sqrt{3}$$

$$= -\cancel{2}\left(\frac{\sqrt{3}}{\cancel{2}}\right) + C = 4\sqrt{3}$$

$$-\sqrt{3} + C = 4\sqrt{3}$$

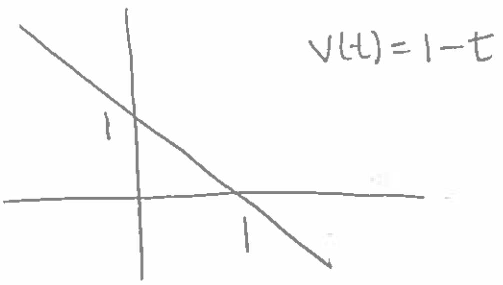
$$\Rightarrow C = 5\sqrt{3}$$

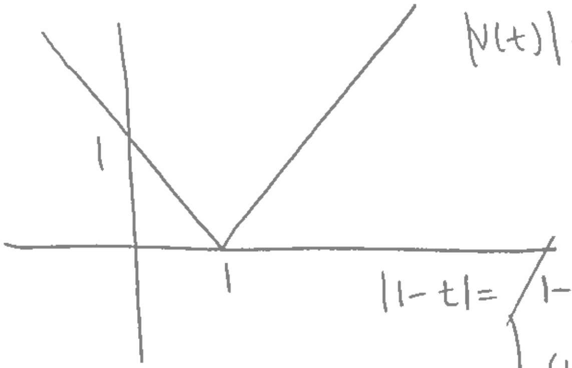


Finally, $\boxed{f(x) = -2\cos\left(\frac{x}{2}\right) + 5\sqrt{3}}$

6. [10 Points] Consider an object travelling with velocity $v(t) = 1 - t$ meters per second.

(a) Sketch and label both $v(t)$ and $|v(t)|$.





$|V(t)| = |1-t|$

$$|1-t| = \begin{cases} 1-t \text{ if } 1-t \geq 0 \Rightarrow t \leq 1 \\ -(1-t) \text{ if } 1-t < 0 \Rightarrow t > 1 \end{cases}$$

(b) Compute the **total distance** travelled by the object from time $t = 0$ to $t = 3$.

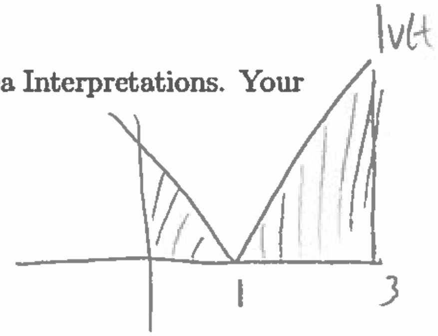
$$\text{Total Distance} = \int_0^3 |v(t)|dt = \int_0^3 |1-t|dt = \int_0^1 \underset{1-t}{\cancel{|1-t|}}dt + \int_1^3 \underset{-(1-t)}{\cancel{|1-t|}}dt$$

$$= \int_0^1 1-t\,dt + \int_1^3 t-1\,dt = t-\frac{t^2}{2}\Big|_0^1 + \frac{t^2}{2} - t\Big|_1^3$$

$$= \left(1-\frac{1}{2}\right) - \cancel{(0-0)} + \left(\frac{9}{2}-3\right) - \overset{-1/2}{\left(\frac{1}{2}-1\right)}$$

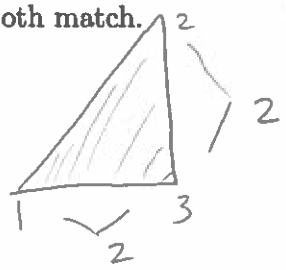
$$= \frac{1}{2} + \frac{9}{2} - 3 + \frac{1}{2} = 1 + \frac{9}{2} - 3 = \frac{9}{2} - 2 = \frac{9}{2} - \frac{4}{2} = \boxed{\frac{5}{2}}$$

(c) Compute the Total Distance Integral from part (b) (this time) using Area Interpretations. Your answers from (b) and (c) should both match.



$\int_0^3 |v(t)|\,dt =$

$+$



$$= \underbrace{\frac{1}{2}\cdot b\cdot h}_{\frac{1}{2}} + \underbrace{\frac{1}{2}\,b\cdot h}_{\frac{1}{2}\cdot 2\cdot 2} = \frac{1}{2} + 2 = \boxed{\frac{5}{2}}$$ Match