Name: Answer Key

Amherst College

DEPARTMENT OF MATHEMATICS

Math 106

Midterm Exam #2

March 22, 2017

- This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, webpages, or other aids are permitted.

- Simplify numerical answers such as $\sin\left(\frac{\pi}{6}\right)$ and $4^{\frac{3}{2}}$.

- Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

Problem	Score	Possible Points
1		22
2		4
3		32
4		32
5		10
Total		100

1. [22 Points] Compute $\int_{-1}^{3} 4 - 3x - x^2\, dx$ using two different methods:

(a) Fundamental Theorem of Calculus

$$\int_{-1}^{3} 4-3x-x^2 dx = 4x - \frac{3x^2}{2} - \frac{x^3}{3}\Big|_{-1}^{3} = 12 - \frac{27}{2} - \cancel{\frac{27}{3}}^{9} - \left(-4 - \frac{3}{2} + \frac{1}{3}\right)$$

$$= 12 - \frac{27}{2} - 9 + 4 + \frac{3}{2} - \frac{1}{3} = 7 - \cancel{\frac{24}{2}}^{12} - \frac{1}{3} = -5 - \frac{1}{3} = \boxed{-\frac{16}{3}}$$

(b) Limit Definition of the Definite Integral.

$f(x) = 4-3x-x^2$

$a = -1$

$b = 3$

$\Delta x = \frac{b-a}{n} = \frac{3-(-1)}{n}$

$x_i = a + i\Delta x$

$= -1 + \frac{4i}{n}$

$$\int_{-1}^{3} 4-3x-x^2 dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f(-1 + 4i/n)\cdot\frac{4}{n}$$

$$= \lim_{n\to\infty} \frac{4}{n} \sum_{i=1}^{n} 4 - 3(-1 + 4i/n) - (-1 + 4i/n)^2$$

$(-1 + 4i/n)^2$

$= 1 - 8i/n + \frac{16i^2}{n^2}$

$$= \lim_{n\to\infty} \frac{4}{n} \sum_{i=1}^{n} 4 + 3 - 12i/n - 1 + \frac{8i}{n} - \frac{16i^2}{n^2}$$

$$= \lim_{n\to\infty} \frac{4}{n} \sum_{i=1}^{n} 6 - 4i/n - \frac{16i^2}{n^2}$$

$$= \lim_{n\to\infty} \frac{4}{n} \sum_{i=1}^{n} 6 - \frac{4}{n} \sum_{i=1}^{n} \frac{4i}{n} - \frac{4}{n} \sum_{i=1}^{n} \frac{16i^2}{n^2}$$

$$= \lim_{n\to\infty} \frac{24}{n} \sum_{i=1}^{n} 1 - \frac{16}{n^2} \sum_{i=1}^{n} i - \frac{64}{n^3} \sum_{i=1}^{n} i^2$$

$$= \lim_{n\to\infty} \cancel{\frac{24}{n}(n)} - \frac{16}{n^2}\left[\frac{n(n+1)}{2}\right] - \frac{64}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty} 24 - \cancel{\frac{16}{2}}^{8}\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right) - \frac{64}{6}\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)$$

$$= \lim_{n\to\infty} 24 - 8(1)(1 + \cancel{1/n}^{0}) - \frac{64}{6}(1)(1 + \cancel{1/n}^{0})(2 + \cancel{1/n}^{0})$$

don't drop

$$= 24 - 8 - \frac{64}{3} = 16 - \frac{64}{3} = \frac{48}{3} - \frac{64}{3} = \boxed{\frac{-16}{3}}$$

Match!

2. [4 Points] Compute $g'(x)$ where $g(x) = \int_x^3 \frac{\sqrt{1+t}}{7+\sec^2 t}\,dt$.

$$g'(x) = \frac{d}{dx}\int_x^3 \frac{\sqrt{1+t}}{7+\sec^2 t}\,dt$$

$$= -\frac{d}{dx}\int_3^x \frac{\sqrt{1+t}}{7+\sec^2 t}\,dt$$

$$= \boxed{-\frac{\sqrt{1+x}}{7+\sec^2 x}}$$ FTC Part I.

3. [32 Points] Evaluate each of the following integrals. Simplify.

(a) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sin x}{\cos^3 x}\,dx = -\int_{\frac{\sqrt{3}}{2}}^{\frac{\sqrt{2}}{2}} \frac{1}{u^3}\,du = -\int_{\frac{\sqrt{3}}{2}}^{\frac{\sqrt{2}}{2}} u^{-3}\,du = -\frac{u^{-2}}{(-2)}\Bigg|_{\sqrt{3}/2}^{\sqrt{2}/2}$

$$\boxed{\begin{array}{l} u = \cos x \\ du = -\sin x\,dx \\ -du = \sin x\,dx \end{array}}$$

$$= \frac{1}{2u^2}\Bigg|_{\sqrt{3}/2}^{\sqrt{2}/2} = \frac{1}{2}\cdot\frac{1}{\left(\frac{\sqrt{2}}{2}\right)^2} - \frac{1}{2}\cdot\frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$\boxed{\begin{array}{l} x = \pi/6 \Rightarrow u = \cos \pi/6 = \frac{\sqrt{3}}{2} \\ x = \pi/4 \Rightarrow u = \cos \pi/4 = \frac{\sqrt{2}}{2} \end{array}}$$

$$= \frac{1}{\cancel{2\cdot\left(\frac{2}{4}\right)}} - \frac{1}{\cancel{2\cdot 3/4}} = 1 - \frac{1}{\left(3/2\right)} = 1 - 2/3 = \boxed{\frac{1}{3}}$$

(b) $\displaystyle\int_{1}^{4} \frac{1}{\sqrt{x}\,(3+\sqrt{x}\,)^2}\,dx = 2\int_4^5 \frac{1}{u^2}\,du = 2\int_4^5 u^{-2}\,du = \frac{2u^{-1}}{-1}\Bigg|_4^5$

$$\boxed{\begin{array}{l} u = 3+\sqrt{x} \\ du = \frac{1}{2\sqrt{x}}\,dx \\ 2du = \frac{1}{\sqrt{x}}\,dx \end{array}}$$

$$= -\frac{2}{u}\Bigg|_4^5 = \frac{-2}{5} + \frac{2}{4} = \frac{-2}{5} + \frac{1}{2}$$

$$\boxed{\begin{array}{l} x = 1 \Rightarrow u = 3 + \sqrt{1} = 4 \\ x = 4 \Rightarrow u = 3 + \sqrt{4} = 5 \end{array}}$$

$$= \frac{-4}{10} + \frac{5}{10} = \boxed{\frac{1}{10}}$$

3. (Continued) Evaluate each of the following integrals. Simplify.

(c) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \tan^3 x \sec^2 x \, dx = \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} u^3 du = \frac{u^4}{4}\Big|_{1/\sqrt{3}}^{\sqrt{3}} = \frac{(\sqrt{3})^4}{4} - \frac{\left(\frac{1}{\sqrt{3}}\right)^4}{4}$

$$\boxed{\begin{array}{l} u = \tan x \\ du = \sec^2 x \, dx \end{array}}$$

$$= \frac{3^2}{4} - \frac{1}{4}\cdot\left(\frac{1}{3}\right)^2 = \frac{9}{4} - \frac{1}{4\cdot 9}$$

$$\boxed{\begin{array}{l} x = \pi/6 \Rightarrow u = \tan \pi/6 = \frac{1}{\sqrt{3}} \\ x = \pi/3 \Rightarrow u = \tan \pi/3 = \sqrt{3} \end{array}}$$

$$= 9/4 - \frac{1}{36} = \frac{81}{36} - \frac{1}{36} = \frac{80}{36} = \boxed{\frac{20}{9}}$$

(d) $\int_0^{\frac{\pi}{3}} \sec x \tan x \sqrt{8 - 4\sec x} \, dx = \frac{-1}{4}\int_4^0 \overset{u^{1/2}}{\sqrt{u}} \, du$

$$\boxed{\begin{array}{l} u = 8 - 4\sec x \\ du = -4\sec x \tan x \, dx \\ -\frac{1}{4} du = \sec x \tan x \, dx \end{array}}$$

$$= \frac{-1}{4}\left(\frac{u^{3/2}}{(3/2)}\right)\Big|_4^0$$

$$= -\frac{1}{\cancel{4}} \cdot \frac{\cancel{2}}{3} u^{3/2}\Big|_4^0$$

$$\boxed{\begin{array}{l} x = 0 \Rightarrow u = 8 - 4\overset{1}{\sec 0} = 4 \\ x = \pi/3 \Rightarrow u = 8 - 4\sec \pi/3 \\ \quad = 8 - \frac{4}{\cos \pi/3} \\ \quad = 8 - \frac{4}{1/2} \\ \quad = 8 - 8 = 0 \end{array}}$$

$$= -1/6\left[\overset{0}{\cancel{0^{3/2}}} - 4^{3/2}\right]$$

$$= \cancel{-}1/6\left[\cancel{-}(\sqrt{4})^3\right] = +\frac{1}{6}(2)^3 = \frac{8}{6} = \boxed{\frac{4}{3}}$$

4. [32 Points] Evaluate each of the following integrals. Simplify.

(a) $\displaystyle\int_1^9 \frac{(1+x^2)(1-\sqrt{x})}{x^2}\,dx = \int_1^9 \frac{1+x^2-\sqrt{x}-x^{5/2}}{x^2}\,dx$

Algebra

$$= \int_1^9 \frac{1}{x^2} + \cancel{\frac{x^2}{x^2}}^{1} - \frac{\sqrt{x}}{x^2} - \frac{x^{5/2}}{x^2}\,dx$$

$$= \int_1^9 x^{-2} + 1 - x^{-3/2} - x^{1/2}\,dx$$

$9^{3/2} = (\sqrt{9})^3 = 3^3$

$$= \frac{x^{-1}}{-1} + x - \frac{x^{-1/2}}{(-1/2)} - \frac{x^{3/2}}{(3/2)}\Big|_1^9 = -1/x + x + \frac{2}{\sqrt{x}} - \frac{2}{3}x^{3/2}\Big|_1^9$$

$$= \frac{-1}{9} + 9 + \frac{2}{\sqrt{9}_{\,3}} - \frac{2}{3}(9)^{3/2} - (\cancel{-1+1} + 2 - \frac{2}{3}) = -1/9 + 9 + \frac{2}{3} - \frac{2}{\cancel{3}}\overset{(9)}{(\cancel{27})} - 2 + 2/3$$

$$= -1/9 + 9 + \frac{2}{3} - 18 - 2 + 2/3 = -1/9 - 11 + 4/3 = -1/9 - \frac{99}{9} + \frac{12}{9} = \frac{-100}{9} + \frac{12}{9} = \boxed{\frac{-88}{9}}$$

(b) $\displaystyle\int \frac{6}{x^3\sqrt{1+\frac{6}{x^2}}}\,dx = \cancel{\frac{-1}{12}\int\frac{6}{\sqrt{u}}\,du}$

$$\boxed{\begin{aligned} u &= 1 + 6/x^2 \\ du &= \frac{-12}{x^3}\,dx \\ \frac{-1}{12}\,du &= \frac{1}{x^3}\,dx \end{aligned}}$$

$$= -1/2 \int u^{-1/2}\,du$$

$$= \cancel{-1/2\,\frac{u^{1/2}}{(1/2)}} + C$$

$$= -\sqrt{u} + C$$

$$= \boxed{-\sqrt{1+6/x^2} + C}$$

4. (Continued) Evaluate each of the following integrals. Simplify.

Algebra now

(c) $\int x\,(8-x)^{\frac{1}{3}}\,dx = -\int (8-u)\,u^{1/3}\,du$

Invert

$u = 8-x \Rightarrow x = 8-u$

$du = -dx$

$-du = dx$

$= -\int 8u^{1/3} - u^{4/3}\,du$

$= -\left[\dfrac{8u^{4/3}}{(4/3)} - \dfrac{u^{7/3}}{(7/3)}\right] + C$

$= -\left[\dfrac{24}{4}u^{4/3} - \dfrac{3}{7}u^{7/3}\right] + C$

$= \boxed{-6(8-x)^{4/3} + \dfrac{3}{7}(8-x)^{7/3} + C}$

(d) $\int x^6\,(8-x^7)^5\,dx = \dfrac{-1}{7}\int u^5\,du = \dfrac{-1}{7}\left(\dfrac{u^6}{6}\right) + C$

$u = 8-x^7$

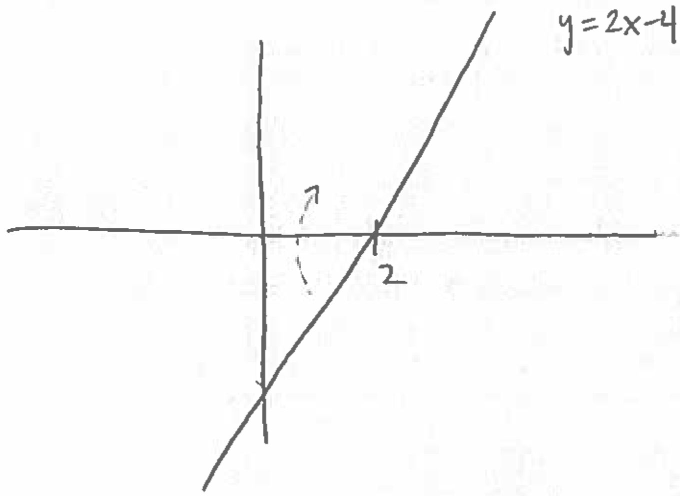
$du = -7x^6\,dx$

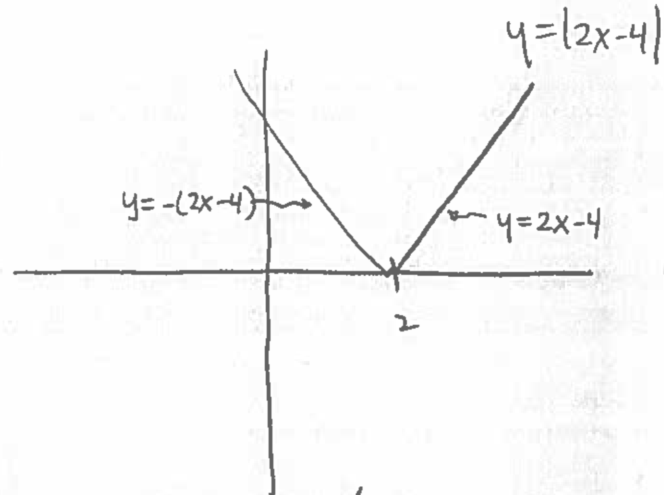
$\dfrac{-1}{7}du = x^6\,dx$

$= \dfrac{-u^6}{42} + C$

$= \boxed{\dfrac{-(8-x^7)^6}{42} + C}$

5. [10 Points] Compute $\int_{-1}^{4} |2x-4| - 1\,dx$





Definition: $|2x-4| = \begin{cases} 2x-4 & \text{if } x \geq 2 \\ -(2x-4) & \text{if } x < 2 \end{cases}$

$$\int_{-1}^{4} |2x-4| - 1\,dx = \int_{-1}^{2} \overset{-(2x-4)}{\cancel{|2x-4|}} - 1\,dx + \int_{2}^{4} \overset{2x-4}{\cancel{|2x-4|}} - 1\,dx$$

$$= \int_{-1}^{2} -2x+4-1\,dx + \int_{2}^{4} 2x-4-1\,dx$$

$$= \int_{-1}^{2} -2x+3\,dx + \int_{2}^{4} 2x-5\,dx$$

$$= -x^2+3x \Big|_{-1}^{2} + x^2-5x \Big|_{2}^{4}$$

$$= -4+6-\underbrace{(-1-3)}_{-4} + 16-20-\overbrace{(4-10)}^{-6}$$

$$= +2+\cancel{4}-\cancel{4}+6$$

$$= \boxed{8}$$