Name: Answer Key

Amherst College

DEPARTMENT OF MATHEMATICS

Math 106

Midterm Exam #1

February 16, 2018

• This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, webpages, or other aids are permitted.

• Simplify numerical answers such as $\sin\left(\frac{\pi}{6}\right)$ and $4^{\frac{3}{2}}$.

• Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

Problem	Score	Possible Points
1		22
2		15
3		20
4		15
5		12
6		16
Total		100

1. [22 Points] Differentiate each of the following functions. **Do not** simplify your answers.

-1 constant

(a) $f(x) = \cancel{\cos}\,\pi + \sqrt{\cos\sqrt{x}}$

$$f'(x) = 0 + \boxed{\frac{1}{2\sqrt{\cos\sqrt{x}}} \cdot \left(-\sin\sqrt{x}\right) \cdot \frac{1}{2\sqrt{x}}}$$

(b) $f(x) = \cos(\sin x)$

$$f'(x) = \boxed{-\sin(\sin x) \cdot \cos x}$$

(c) $f(x) = \cos x \cdot \sin x$

$$f'(x) = \cos x \cdot \cos x + \sin x(-\sin x) = \boxed{\cos^2 x - \sin^2 x}$$

(d) $f(x) = \cos^5\left(\frac{7}{x^6}\right) = \left[\cos\left(7x^{-6}\right)\right]^5$

$$f'(x) = \boxed{5\left[\cos\left(7x^{-6}\right)\right]^4 \cdot \left(-\sin\left(7x^{-6}\right)\right) \cdot \left(-42x^{-7}\right)}$$

(e) $f(x) = \left(\frac{\cos(7x)}{\tan(3x)}\right)^{\frac{7}{8}}$

$$f'(x) = \frac{7}{8}\left(\frac{\cos(7x)}{\tan(3x)}\right)^{-1/8}\left[\frac{\tan(3x)\left(-\sin(7x)\right)(7) - \cos(7x)\sec^2(3x)\cdot 3}{\left(\tan(3x)\right)^2}\right]$$

2. [15 Points] Derivatives

(a) Let $f(x) = \dfrac{1}{\sin^2 x} + \tan(2x)\cdot\sin(2x)$. Compute $f'\left(\frac{\pi}{6}\right)$. Simplify.

Product

$$= (\sin x)^{-2} + \tan(2x)\cdot\sin(2x)$$

$$f'(x) = -2(\sin x)^{-3}\cdot\cos x + \tan(2x)\cos(2x)\cdot 2 + \sin(2x)\sec^2(2x)\cdot 2$$

$$f'(\pi/6) = \frac{-2}{\left(\sin(\pi/6)\right)^3}\cdot\cos(\pi/6) + \tan\left(\frac{\pi}{3}\right)\cos(\pi/3)\cdot 2 + \sin(\pi/3)\sec^2(\pi/3)\cdot 2$$

$2\cdot(\pi/6)$

$\dfrac{1}{\cos^2(\pi/3)}$

$$= \frac{-\cancel{2}}{(1/2)^3}\cdot\left(\frac{\sqrt{3}}{\cancel{2}}\right) + \sqrt{3}\left(\frac{1}{\cancel{2}}\right)\cdot\cancel{2} + \frac{\sqrt{3}}{\cancel{2}}\cdot\frac{1}{(1/2)^2}\cdot\cancel{2}$$

$$= -8\sqrt{3} + \sqrt{3} + 4\sqrt{3}$$

$$= -8\sqrt{3} + 5\sqrt{3}$$

$$= \boxed{-3\sqrt{3}}$$

(b) Let $f(x) = 4\sin\left(x - \frac{\pi}{4}\right) - \cos x - \tan^2 x$. Show that $f'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$.

$$f'(x) = 4\cos\left(x - \pi/4\right)(1) + \sin x - 2\tan x\cdot\sec^2 x$$

$$f'(\pi/4) = 4\cos\cancel{0} + \sin\cancel{(\pi/4)} - 2\tan\cancel{(\pi/4)}\cdot\sec^2(\pi/4)$$

$1 \qquad \sqrt{2}/2 \qquad 1 \qquad (\sqrt{2})^2$

$$= \cancel{4} + \frac{\sqrt{2}}{2} - \cancel{4} = \boxed{\frac{\sqrt{2}}{2}} \quad \text{Match!}$$

$$\sec \pi/4 = \frac{1}{\cos(\pi/4)} = \frac{1}{\left(1/\sqrt{2}\right)} = \sqrt{2}$$

3. [20 Points] Compute the following **most general antiderivatives.** Do not simplify your final answer.

(a) $\int \frac{5}{6}x + x^{\frac{5}{6}} + \frac{1}{x^{\frac{5}{6}}} + \frac{6}{5} + \frac{5}{6x^6} - \frac{6}{x^5}\,dx$ $= \int \frac{5}{6}x + x^{5/6} + x^{-5/6} + \frac{6}{5} + \frac{5}{6}x^{-6} - 6x^{-5}\,dx$

$= \frac{5}{6}\left(\frac{x^2}{2}\right) + \frac{x^{11/6}}{(11/6)} + \frac{x^{1/6}}{(1/6)} + \frac{6}{5}x + \frac{5}{6}\left(\frac{x^{-5}}{-5}\right) - \frac{6x^{-4}}{-4} + C$

$= \boxed{\frac{5x^2}{12} + \frac{6}{11}x^{11/6} + 6x^{1/6} + \frac{6}{5}x - \frac{1}{6x^5} + \frac{6}{4x^4} + C}$

(b) $\int \sec^2 x - 8\cos x + \sin x + \frac{\sec x \tan x}{7}\,dx$

$= \boxed{\tan x - 8\sin x - \cos x + \frac{1}{7}\cdot \sec x + C}$

3. (Continued) Compute the following **most general antiderivatives.** Do not simplify your final answer.

(c) $\displaystyle\int \left(x^2+\frac{1}{x^2}\right)\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)\,dx = \int x^2\sqrt{x} + \frac{x^2}{\sqrt{x}} + \frac{\sqrt{x}}{x^2} + \frac{1}{x^2\cdot\sqrt{x}}\,dx$

$$= \int x^{5/2} + x^{3/2} + x^{-3/2} + x^{-5/2}\,dx = \frac{x^{7/2}}{7/2} + \frac{x^{5/2}}{5/2} + \frac{x^{-1/2}}{-1/2} + \frac{x^{-3/2}}{-3/2} + C$$

$$= \boxed{\frac{2}{7}x^{7/2} + \frac{2}{5}x^{5/2} - \frac{2}{\sqrt{x}} - \frac{2}{3x^{3/2}} + C}$$

$\frac{7}{14} - \frac{6}{14}$

SPLIT $\quad \frac{14}{7} \quad x^{\frac{1}{2}-\frac{3}{7}}$

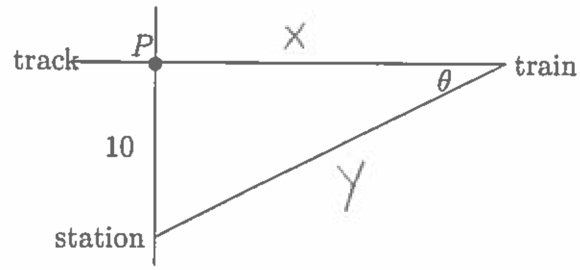
(d) $\displaystyle\int \frac{x^2+\sqrt{x}}{x^{\frac{3}{7}}}\,dx = \int \frac{x^2}{x^{3/7}} + \frac{\sqrt{x}}{x^{3/7}}\,dx = \int x^{11/7} + x^{1/14}\,dx$

$$= \frac{x^{18/7}}{\left(18/7\right)} + \frac{x^{15/14}}{\left(15/14\right)} + C = \boxed{\frac{7}{18}x^{18/7} + \frac{14}{15}x^{15/14} + C}$$

4. [15 Points] Consider a point P on a train track. Suppose a train depot station is 10 feet directly south from this point P. The train is travelling east at 6 feet per second. Consider the angle as shown in the diagram. How fast is this angle changing when 2 seconds has passed since the train passed point P.

- Diagram

The picture at arbitrary time t is:



- Variables

Let X = distance the train has travelled East past Point P.

Y = distance between train and station

θ = angle between the track and line connecting train + station

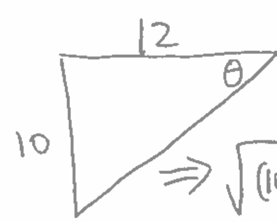
- Given $\frac{dx}{dt} = 6$ ft/sec. Find $\frac{d\theta}{dt} = ?$ when $x = 6$ ft./sec. (2 sec.) = 12 ft.
- Equation:

$$\tan\theta = \frac{10}{X} \rightarrow 10X^{-1}$$

- Differentiate

$$\sec^2\theta \frac{d\theta}{dt} = -10X^{-2} \cdot \frac{dx}{dt}$$

- Extra Solvable Information.



$\Rightarrow \sqrt{(10)^2+(12)^2}$

$= \sqrt{100+144}$

$= \sqrt{244}$

$\Rightarrow \sec\theta = \frac{H}{A} = \frac{\sqrt{244}}{12}$

- Substitute

$$\left(\frac{\sqrt{244}}{12}\right)^2 \frac{d\theta}{dt} = \frac{-10}{(12)^2} \cdot (6)$$

- Solve

$$\frac{d\theta}{dt} = \frac{-60}{\cancel{(12)^2}} \cdot \frac{\cancel{(12)^2}}{244} = \frac{-60}{244} = \boxed{\frac{-15}{61}} \text{ Rad./sec.}$$

- Answer

The angle is shrinking at a rate of $15/61$ Radians per second at that moment.

5. [12 Points] Consider a function f such that $f''(x) = \pi \sin x + 2\cos x$ and $f'\left(\frac{\pi}{2}\right) = 0$ and $f(\pi) = 2$. Compute $f(x)$.

$$f''(x) = \pi \sin x + 2\cos x$$

$$\int$$

$$f'(x) = -\pi \cos x + 2\sin x + C_1$$

$$f'(\pi/2) = -\pi \underbrace{\cancel{\cos(\pi/2)}}_{0} + 2\overbrace{\cancel{\sin(\pi/2)}}^{1} + C_1 \overset{\text{set}}{=} 0$$

$$0 + 2 + C_1 = 0$$

$$\text{Solve } C_1 = -2$$

Now, $f'(x) = -\pi \cos x + 2\sin x - 2$

$$\int$$

$$f(x) = -\pi \sin x - 2\cos x - 2x + C_2$$

$$f(\pi) = -\pi \underbrace{\cancel{\sin \pi}}_{0} - 2\underbrace{\cancel{\cos \pi}}_{-1} - 2\pi + C_2 \overset{\text{set}}{=} 2$$

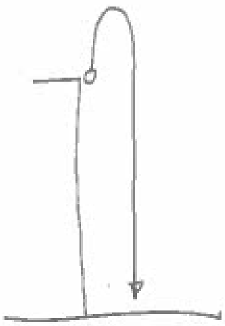
$$\cancel{+2} - 2\pi + C_2 = \cancel{2}$$

$$\text{Solve } C_2 = 2\pi$$

Finally, $f(x) = \boxed{-\pi \sin x - 2\cos x - 2x + 2\pi}$

6. [16 Points] For each of the following use $a(t) = -32$ feet per second squared as acceleration due to gravity on the falling body.

(a) A ball is thrown upwards from the top of a building with an initial *speed* of 32 feet per second. The ball hits the ground below with a *speed* of 64 feet per second. How tall is the building?



Equations:

$a(t) = -32$

$v(t) = -32t + V_0$

$= -32t + 32$

$s(t) = -16t^2 + V_0 t + S_0$

$= -16t^2 + 32t + S_0$

?

$V_0 = +32$ ft/sec

$S_0 = ?$

$V(t_{impact}) = -64$ ft/sec

Down

Use Velocity @ impact to find Time at impact.

$V(t) = -32t + 32 \overset{set}{=} -64$

$\frac{-32t}{-32} = \frac{-96}{-32}$

$t_{impact} = 3$ seconds

Now use $S(t_{impact}) \overset{set}{=} 0$ to solve for S_0 = initial position

$s(t) = -16(3)^2 + 32(3) + S_0 \overset{set}{=} 0$

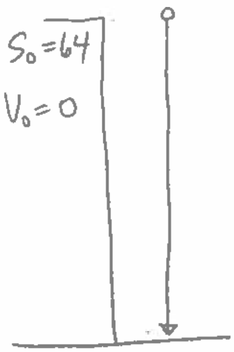
$-144 + 96 + S_0 = 0$

Solve $S_0 = 144 - 96 = \boxed{48 \text{ feet}}$

Answer: The building was 48 feet tall.

6. (Continued) For each of the following use $a(t) = -32$ feet per second squared as acceleration due to gravity on the falling body.

(b) A ball is dropped straight down from the top of a building 64 feet tall, with intitial velocity of 0 feet per second. What is the velocity at impact?



Equations:

$$a(t) = -32$$

$$V(t) = -32t + \cancel{v_0}^{\,0}$$

$$= -32t$$

$$s(t) = -16t^2 + \cancel{v_0 t}^{\,0} + \cancel{s_0}^{\,64}$$

$$= -16t^2 + 64$$

Need time at impact. Set $s(t) = 0$

$$s(t) = -16t^2 + 64 \overset{set}{=} 0$$

$$16t^2 = 64$$

$$t^2 = 4$$

$$t_{impact} = \cancel{\pm}2 \quad \text{(positive time)}$$

Velocity at impact, $t=2$ seconds

$$V(2) = -32(2) = \boxed{-64 \text{ ft/sec.}}$$

Answer: The velocity at impact is -64 ft./sec. or 64 ft/sec. Down.