Answer Key

Math 106 **Midterm Exam #1** **February 19, 2014**

1. [20 Points] Differentiate each of the following functions. **Do not** simplify your answers.

(a) $f(x) = \cos\left(\tan^2\left(\frac{3}{x^5}\right)\right) + \tan\left(\cos^2\left(\frac{x^5}{3}\right)\right)$

$f'(x) = \boxed{-\sin\left(\tan^2\left(\frac{3}{x^5}\right)\right) \cdot 2\tan\left(\frac{3}{x^5}\right) \cdot \sec^2\left(\frac{3}{x^5}\right)\left(-\frac{15}{x^6}\right) + \ldots}$ (continued below)

$\boxed{\ldots \sec^2\left(\cos^2\left(\frac{x^5}{3}\right)\right) \cdot 2\cos\left(\frac{x^5}{3}\right) \cdot \left(-\sin\left(\frac{x^5}{3}\right)\right)\left(\frac{5}{3}x^4\right)}$

(b) $y = \sec^2(7x) \cdot \tan(7x^2)$

$y' = \boxed{\sec^2(7x)\sec^2(7x^2)(14x) + \tan(7x^2)2\sec(7x)\sec(7x)\tan(7x)(7)}$

(c) $g(t) = \dfrac{\sqrt{t} + \sec(\sqrt{t})}{\sqrt{1+\sec t}}$

$$g'(t) = \boxed{\frac{\sqrt{1+\sec t}\left(\frac{1}{2\sqrt{t}} + \sec(\sqrt{t})\tan(\sqrt{t})\left(\frac{1}{2\sqrt{t}}\right)\right) - \left(\sqrt{t} + \sec(\sqrt{t})\right)\left(\frac{1}{2\sqrt{1+\sec t}}\right)(\sec t \tan t)}{1+\sec t}}$$

2. [15 Points] Consider the curve given by $\cos(xy^2) + 2 = \sin x + y^3$.

(a) Compute $\frac{dy}{dx}$.

First implicitly differentiate both sides:

$\frac{d}{dx}\left(\cos(xy^2) + 2\right) = \frac{d}{dx}\left(\sin x + y^3\right)$

$-\sin(xy^2)\left(x(2y)\frac{dy}{dx} + y^2\right) + 0 = \cos x + 3y^2\frac{dy}{dx}$

Distribute:

$-2xy\sin(xy^2)\frac{dy}{dx} - y^2\sin(xy^2) = \cos x + 3y^2\frac{dy}{dx}$

Isolate and factor $\frac{dy}{dx}$:

$-2xy\sin(xy^2)\frac{dy}{dx} - 3y^2\frac{dy}{dx} = \cos x + y^2\sin(xy^2)$

$\left(-2xy\sin(xy^2) - 3y^2\right)\frac{dy}{dx} = \cos x + y^2\sin(xy^2)$

Solve:

$$\frac{dy}{dx} = \boxed{\frac{\cos x + y^2\sin(xy^2)}{-2xy\sin(xy^2) - 3y^2}}$$

(b) Find the equation of the tangent line to this curve at the point $(\pi, 1)$.

Plug in the point values, $x = \pi$ and $y = 1$

Finally, solve $\left.\frac{dy}{dx}\right|_{(\pi,1)} = \frac{\cos\pi + y^2\sin(0)}{0\sin(0) - 3} = \frac{-1+0}{0-3} = \frac{1}{3}$

Using point-slope form we get the equation of the tangent line:

$y - 1 = \frac{1}{3}(x - \pi)$ or $\boxed{y = \frac{1}{3}x - \frac{\pi}{3} + 1}$

3. [20 Points]

(a) Let $f(x) = \sin^3(4x) + \sec(4x) - 8\sin(2x)$. Compute $f'\left(\frac{\pi}{12}\right)$. Simplify.

Differentiate:

$f'(x) = 3\sin^2(4x)\cos(4x)(4) + 4\sec(4x)\tan(4x) - 16\cos(2x)$

Note: $4\left(\frac{\pi}{12}\right) = \frac{\pi}{3}$ and $2\left(\frac{\pi}{12}\right) = \frac{\pi}{6}$

Evaluate:

$f'\left(\frac{\pi}{12}\right) = 3\sin^2\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right)(4) + 4\sec\left(\frac{\pi}{3}\right)\tan\left(\frac{\pi}{3}\right) - 16\cos\left(\frac{\pi}{6}\right)$

$= 3\left(\frac{\sqrt{3}}{2}\right)^2\left(\frac{1}{2}\right)(4) + 4(2)\sqrt{3} - 16\left(\frac{\sqrt{3}}{2}\right)$

$= 3\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)(4) + 8\sqrt{3} - 8\sqrt{3} = \boxed{\frac{9}{2}}$

(b) Let $f(x) = \cos(2x) + \frac{1}{\tan^2 x} + \sin\left(x - \frac{\pi}{4}\right)$. Compute $f'\left(\frac{\pi}{4}\right)$. Simplify.

Note: $2\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$

Differentiate:

$f'(x) = -2\sin(2x) - 2(\tan x)^{-3}\sec^2 x + \cos\left(x - \frac{\pi}{4}\right)$

$= -2\sin(2x) - \frac{2\sec^2 x}{\tan^3 x} + \cos\left(x - \frac{\pi}{4}\right)$

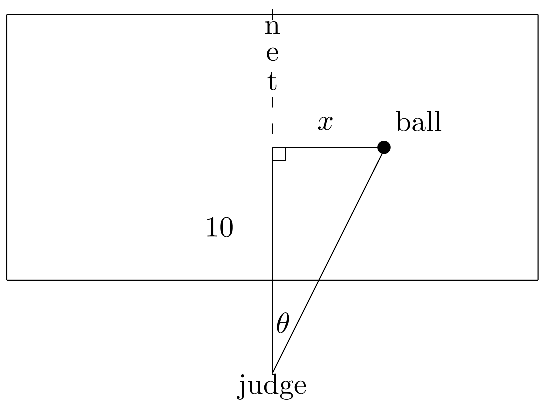
Evaluate:

$$f'\left(\frac{\pi}{4}\right) = -2\sin\left(\frac{\pi}{2}\right) - \frac{2\sec^2\left(\frac{\pi}{4}\right)}{\tan^3\left(\frac{\pi}{4}\right)} + \cos\left(\frac{\pi}{4} - \frac{\pi}{4}\right) = -2\sin\left(\frac{\pi}{2}\right) - \frac{2\sec^2\left(\frac{\pi}{4}\right)}{\tan^3\left(\frac{\pi}{4}\right)} + \cos(0)$$

$$= -2(1) - \frac{2(\sqrt{2})^2}{1^3} + 1 = -2 - 4 + 1 = \boxed{-5}$$

4. [15 Points] A judge is watching a tennis match. She is sitting on the court, in line with the net, 10 meters from the center of the net. The tennis ball is travelling perpendicular to the net on a path that passes directly over the center of the net. How quickly is the judge rotating her head to keep the ball in sight, when the ball is moving at 2 meters per second and is 3 meters from the net?

• Diagram



• Variables

Let x =distance of ball to center of net at time t

Let θ = angle judge turns head from net at time t

Find $\frac{d\theta}{dt}$ =? when $x = 3$ feet

and $\frac{dx}{dt} = 2\frac{\text{m}}{\text{sec}}$

Note this is negative b/c we are fixing driving left to right towards the nearest point.

• Equation relating the variables:

$$\tan\theta = \frac{x}{10}$$

• Differentiate both sides w.r.t. time t.

$$\frac{d}{dt}(\tan\theta) = \frac{d}{dt}\left(\frac{x}{10}\right) \implies \sec^2\theta\frac{d\theta}{dt} = \frac{1}{10}\frac{dx}{dt} \text{ (Related Rates!)}$$

• Extra Solvable Information

We compute that the hypotenuse is

$$\sqrt{x^2 + (10)^2} = \sqrt{3^2 + (10)^2} = \sqrt{9 + 100} = \sqrt{109}$$

so that at the key moment, $\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{109}}{10}$.

• Substitute Key Moment Information (now and not before now!!!):

We have $\left(\frac{\sqrt{109}}{10}\right)^2 \frac{d\theta}{dt} = \frac{1}{10}(2)$

• Solve for the desired quantity: So the rate of change of rotation is

$\frac{d\theta}{dt} = \frac{2}{10} \cdot \frac{100}{109} = \frac{20}{109}$ radians/sec.

• Answer the question that was asked: The judge must be rotating her head at a rate of $\frac{20}{109}$ radians every second.

5. [20 Points] Find the general antiderivative for each of the following:

(a) $\dfrac{x^6 + \frac{1}{\sqrt{x}}}{x^{\frac{3}{4}}}$

Simplify first:

$$= \frac{x^6}{x^{\frac{3}{4}}} + \frac{\frac{1}{\sqrt{x}}}{x^{\frac{3}{4}}} = x^{6-\frac{3}{4}} + x^{-\frac{1}{2}-\frac{3}{4}} = x^{\frac{24}{4}-\frac{3}{4}} + x^{-\frac{2}{4}-\frac{3}{4}} = x^{\frac{21}{4}} + x^{-\frac{5}{4}}$$

Antidifferentiate:

$$\frac{x^{\frac{25}{4}}}{\frac{25}{4}} + \frac{x^{-\frac{1}{4}}}{-\frac{1}{4}} + C = \boxed{\frac{4}{25}x^{\frac{25}{4}} - 4x^{-\frac{1}{4}} + C}$$

(b) $\left(x^{\frac{2}{3}} + \frac{1}{x^2}\right)\left(\sqrt{x} - \frac{1}{x^{\frac{3}{5}}}\right)$

Simplify first:

$$= x^{\frac{2}{3}}x^{\frac{1}{2}} + x^{\frac{1}{2}}x^{-2} - x^{\frac{2}{3}}x^{-\frac{3}{5}} - x^{-2}x^{-\frac{3}{5}}$$

$$= x^{\frac{2}{3}+\frac{1}{2}} + x^{\frac{1}{2}-2} - x^{\frac{2}{3}-\frac{3}{5}} - x^{-2-\frac{3}{5}}$$

$$= x^{\frac{4}{6}+\frac{3}{6}} + x^{\frac{1}{2}-\frac{4}{2}} - x^{\frac{10}{15}-\frac{9}{15}} - x^{-\frac{10}{5}-\frac{3}{5}}$$

$$= x^{\frac{7}{6}} + x^{-\frac{3}{2}} - x^{\frac{1}{15}} - x^{-\frac{13}{5}}$$

Antidifferentiate:

$$\frac{x^{\frac{13}{6}}}{\frac{13}{6}} + \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} - \frac{x^{\frac{16}{15}}}{\frac{16}{15}} - \frac{x^{-\frac{8}{5}}}{-\frac{8}{5}} + C$$

$$= \boxed{\frac{6}{13}x^{\frac{13}{6}} - 2x^{-\frac{1}{2}} - \frac{15}{16}x^{\frac{16}{15}} + \frac{5}{8}x^{-\frac{8}{5}} + C}$$

6. [10 Points] A nearby cave has an increasing population of bats. An agricultural company has been studying the cave since 2000 because guano (bat poop) makes great fertilizer. The company estimates that the the amount of guano, $G(t)$, has been increasing at a *rate* of $G'(t) = 2 + \frac{3}{20}t^2$ tons per year, where t is time in years since 2000. In 2010, ten years later, the company cleaned out the guano and found 80 tons of the stuff. How much guano was in the cave in 2000 at time $t = 0$?

To find the amount $G(t)$, we antidifferentiate:

To start $G'(t) = 2 + \frac{3}{20}t^2$

then $G(t) = 2t + \frac{1}{20}t^3 + C$.

Use the given information $G(10) = 80$ to sovle for C.

Here $G(10) = 2(10) + \frac{1}{20}(10)^3 + C \overset{\text{set}}{=} 80$

$20 + \frac{1000}{20} + C = 80 \quad \Rightarrow 20 + 50 + C = 80 \quad \Rightarrow C = 10$

Finally, $G(t) = 2t + \frac{1}{20}t^3 + 10$. and we solve

$G(0) = 2(0) + \frac{1}{20}(0)^3 + 10 = 10$.

$\boxed{\text{There were 10 tons of guano in the cave in 2000.}}$