Math 106 Final Examination May 13, 2014 Answer Key

1. [30 Points] Compute each of the following derivatives.

(a) $f''(1)$, where $f(x) = \dfrac{\ln x}{x}$

First, $f'(x) = \dfrac{x\left(\frac{1}{x}\right) - \ln x(1)}{x^2} = \dfrac{1-\ln x}{x^2}$.

Next, $f''(x) = \dfrac{x^2\left(-\frac{1}{x}\right) - (1-\ln x)(2x)}{(x^2)^2} = \dfrac{-x-2x(1-\ln x)}{x^4}$.

Finally, $f''(1) = \dfrac{-1-2(1-\ln 1)}{1} = -1-2 = \boxed{-3}$

(b) $\dfrac{d}{dx}\ln\left(\dfrac{e^{3-x}\sqrt{\sec x+\sqrt{x}}}{e^{\ln x}(4-x^4)^4}\right)$ Do not simplify the final answer here.

First simplify using algebraic properties of the log.

$$\frac{d}{dx}\ln\left(\frac{e^{3-x}\sqrt{\sec x+\sqrt{x}}}{e^{\ln x}(4-x^4)^4}\right) = \frac{d}{dx}\ln\left(e^{3-x}\sqrt{\sec x+\sqrt{x}}\right) - \ln\left(e^{\ln x}(4-x^4)^4\right)$$

$$= \frac{d}{dx}\ln\left(e^{3-x}\right) + \ln\left(\sqrt{\sec x+\sqrt{x}}\right) - \ln\left(e^{\ln x}\right) - \ln\left((4-x^4)^4\right)$$

$$= \frac{d}{dx}(3-x) + \frac{1}{2}\ln\left(\sec x+\sqrt{x}\right) - \ln x - 4\ln\left(4-x^4\right)$$

$$= \boxed{-1+\frac{1}{2}\left(\frac{1}{\sec x+\sqrt{x}}\right)\left(\sec x\tan x+\frac{1}{2\sqrt{x}}\right) - \frac{1}{x} - 4\left(\frac{1}{4-x^4}\right)(-4x^3)}$$

(c) $g'(x)$, where $g(x) = (\sin x)^{\frac{1}{x}}$

Let $y = (\sin x)^{\frac{1}{x}}$. (or keep using $g(x)$, it doesn't matter). Looking for $\dfrac{dy}{dx}$.

$\ln y = \ln\left[(\sin x)^{\frac{1}{x}}\right] = \dfrac{1}{x}\ln(\sin x)$

Implicitly differentiate

$\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{1}{x}\dfrac{1}{\sin x}(\cos x) + \ln(\sin x)\left(-\dfrac{1}{x^2}\right)$

Solve $\dfrac{dy}{dx} = y\left(\dfrac{\cos x}{x\sin x} - \dfrac{\ln(\sin x)}{x^2}\right) = \boxed{(\sin x)^{\frac{1}{x}}\left(\dfrac{\cos x}{x\sin x} - \dfrac{\ln(\sin x)}{x^2}\right)}$

(d) $\dfrac{dy}{dx}$, if $\sec^2 y - \cos x = e^{x^2 y} + (\ln(e+5))x$

Implicitly differentiate:

$$\frac{d}{dx}\sec^2 y - \cos x = \frac{d}{dx}e^{x^2y} + (\ln(e+5))x$$

$$2\sec y(\sec y\tan y)\frac{dy}{dx} + \sin x = e^{x^2y}\left(x^2\frac{dy}{dx} + y(2x)\right) + \ln(e+5)$$

Distribute:

$$2\sec y(\sec y\tan y)\frac{dy}{dx} + \sin x = x^2e^{x^2y}\frac{dy}{dx} + 2xye^{x^2y} + \ln(e+5)$$

Isolate $\frac{dy}{dx}$ terms:

$$2\sec y(\sec y\tan y)\frac{dy}{dx} - x^2e^{x^2y}\frac{dy}{dx} = 2xye^{x^2y} + \ln(e+5) - \sin x$$

Factor:

$$\left(2\sec^2 y\tan y - x^2e^{x^2y}\right)\frac{dy}{dx} = 2xye^{x^2y} + \ln(e+5) - \sin x$$

Solve:

$$\frac{dy}{dx} = \boxed{\frac{2xye^{x^2y} + \ln(e+5) - \sin x}{2\sec^2 y\tan y - x^2e^{x^2y}}}$$

(e) y', where $y = \dfrac{1}{\sin\sqrt{e^x+e^7}} + \dfrac{1}{e^{\sqrt{x^2+7\sin x}}} + \dfrac{1}{\sqrt{7\sin x + e^{x^2}}}$ Do not simplify.

First rewrite to simplify and prepare:

$$y = \left(\sin\sqrt{e^x+e^7}\right)^{-1} + e^{-\sqrt{x^2+7\sin x}} + \left(7\sin x + e^{x^2}\right)^{-\frac{1}{2}}$$

$$y' = \boxed{-\left(\sin\sqrt{e^x+e^7}\right)^{-2}\cos\sqrt{e^x+e^7}\left(\frac{1}{2\sqrt{e^x+e^7}}\right)(e^x)} \text{ (continued ...)}$$

$$\boxed{+e^{-\sqrt{x^2+7\sin x}}\left(-\frac{1}{2\sqrt{x^2+7\sin x}}\right)(2x+7\cos x)} \text{ (continued ...)}$$

$$\boxed{-\frac{1}{2}\left(7\sin x + e^{x^2}\right)^{-\frac{3}{2}}\left(7\cos x + e^{x^2}(2x)\right)}$$

2. [20 Points] Compute each of the following derivatives.

(a) $f'\left(\frac{\pi}{6}\right)$ where $f(x) = \sec(2x) + \cos^2(2x) + \dfrac{2}{\sin x}$

First, $f'(x) = 2\sec(2x)\tan(2x) + 2\cos(2x)(-\sin(2x))(2) - 2(\sin x)^{-2}\cos x$.

Next, $f'\left(\frac{\pi}{6}\right) = 2\sec\left(2\left(\frac{\pi}{6}\right)\right)\tan\left(2\left(\frac{\pi}{6}\right)\right) + 2\cos\left(2\left(\frac{\pi}{6}\right)\right)\left(-\sin\left(2\left(\frac{\pi}{6}\right)\right)\right)(2) - \dfrac{2\cos\left(\frac{\pi}{6}\right)}{\left(\sin\left(\frac{\pi}{6}\right)\right)^2}$

$$= 2\sec\left(\frac{\pi}{3}\right)\tan\left(\frac{\pi}{3}\right) - 4\cos\left(\frac{\pi}{3}\right)\left(\sin\left(\frac{\pi}{3}\right)\right) - \frac{2\cos\left(\frac{\pi}{6}\right)}{\left(\sin\left(\frac{\pi}{6}\right)\right)^2}$$

$$= 2(2)\sqrt{3} - 4\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \frac{2\left(\frac{\sqrt{3}}{2}\right)}{\left(\frac{1}{2}\right)^2} = 4\sqrt{3} - \sqrt{3} - 4\sqrt{3} = \boxed{-\sqrt{3}}$$

(b) $f'\left(\frac{\pi}{3}\right)$ where $f(x) = \cos(3x - \pi) + \sec\left(\frac{\pi}{3}\right) + 3\sin\left(x - \frac{\pi}{3}\right)$

First, $f'(x) = -\sin(3x - \pi)(3) + 0 + 3\cos\left(x - \frac{\pi}{3}\right)$

Then, $f'\left(\frac{\pi}{3}\right) = -3\sin\left(3\left(\frac{\pi}{3}\right) - \pi\right) + 3\cos\left(\left(\frac{\pi}{3}\right) - \frac{\pi}{3}\right) = -3\sin 0 + 3\cos 0 = 0 + 3 = \boxed{3}$

(c) $f'\left(\frac{\pi}{4}\right)$ where $f(x) = \cos(2x) - \sin(2x) + 2\tan x$

First, $f'(x) = -\sin(2x)(2) - \cos(2x)(2) + 2\sec^2 x$

Next, $f'\left(\frac{\pi}{4}\right) = -\sin\left(2\left(\frac{\pi}{4}\right)\right)(2) - \cos\left(2\left(\frac{\pi}{4}\right)\right)(2) + 2\sec^2\left(\frac{\pi}{4}\right)$

$= -2\sin\left(\frac{\pi}{2}\right) - 2\cos\left(\frac{\pi}{2}\right) + 2\sec^2\left(\frac{\pi}{4}\right) = -2(1) - 2(0) + 2(\sqrt{2})^2 = -2 + 4 = \boxed{2}$

3. [35 Points] Compute each of the following integrals.

(a) $$\int \frac{(1-\sqrt{x})(x^{\frac{1}{3}} - x)}{x^2}\,dx = \int \frac{x^{\frac{1}{3}} - x^{\frac{5}{6}} - x + x^{\frac{3}{2}}}{x^2}\,dx = \int \frac{x^{\frac{1}{3}}}{x^2} - \frac{x^{\frac{5}{6}}}{x^2} - \frac{x}{x^2} + \frac{x^{\frac{3}{2}}}{x^2}\,dx$$

$$= \int x^{-\frac{5}{3}} - x^{-\frac{7}{6}} - \frac{1}{x} + x^{-\frac{1}{2}}\,dx = \boxed{-\frac{3}{2}x^{-\frac{2}{3}} + 6x^{-\frac{1}{6}} - \ln|x| + 2\sqrt{x} + C}$$

(b) $$\int x\,(3x-1)^{2014}\,dx = \frac{1}{3}\int \left(\frac{u+1}{3}\right)u^{2014}\,du = \frac{1}{9}\int u^{2015} + u^{2014}\,du$$

$$= \frac{1}{9}\left(\frac{u^{2016}}{2016} + \frac{u^{2015}}{2015}\right) + C = \boxed{\frac{1}{9}\left(\frac{(3x-1)^{2016}}{2016} + \frac{(3x-1)^{2015}}{2015}\right) + C}$$

Here $$\boxed{\begin{aligned} u &= 3x - 1 \Rightarrow x = \frac{u+1}{3} \\ du &= 3dx \\ \tfrac{1}{3}du &= dx \end{aligned}}$$

(c) $$\int_1^2 \frac{x}{x^2-9}\,dx = \frac{1}{2}\int_{-8}^{-5} \frac{1}{u}\,du = \frac{1}{2}\ln|u|\Big|_{-8}^{-5} = \frac{1}{2}(\ln|-5| - \ln|-8|) = \boxed{\ln\left(\frac{5}{8}\right)}$$

Here $\boxed{\begin{aligned} u &= x^2 - 9 \\ du &= 2x\,dx \\ \tfrac{1}{2}du &= x\,dx \end{aligned}}$ and $\boxed{\begin{aligned} x = 1 &\implies u = -8 \\ x = 2 &\implies u = -5 \end{aligned}}$

(d) $\displaystyle\int \frac{1}{x^2 e^{\frac{1}{x}}}\,dx = -\int \frac{1}{e^u}\,du = -\int e^{-u}\,du = -\left(-e^{-u}\right) + C = e^{-u} + C = \boxed{\frac{1}{e^{\frac{1}{x}}} + C}$

Here $\boxed{\begin{aligned} u &= \frac{1}{x} \\ du &= -\frac{1}{x^2}dx \\ -du &= \frac{1}{x^2}dx \end{aligned}}$

(e) $\displaystyle\int_{e^3}^{e^8} \frac{5}{x\sqrt{1+\ln x}}\,dx = \int_4^9 \frac{5}{\sqrt{u}}\,du = 5\int_4^9 u^{-\frac{1}{2}}\,du = 10\sqrt{u}\Big|_4^9 = 10(\sqrt{9} - \sqrt{4}) = 10(3-2) = \boxed{10}$

Here $\boxed{\begin{aligned} u &= 1 + \ln x \\ du &\ \frac{1}{x}dx \end{aligned}}$ and $\boxed{\begin{aligned} x = e^3 &\implies u = 1 + \ln e^3 = 1 + 3 = 4 \\ x = e^8 &\implies u = 1 + \ln e^8 = 1 + 8 = 9 \end{aligned}}$

(f) $\displaystyle\int_2^6 \frac{\sin\left(\frac{\pi}{x}\right)\cos\left(\frac{\pi}{x}\right)}{x^2}\,dx = -\frac{1}{\pi}\int_1^{\frac{1}{2}} u\,du = -\frac{1}{\pi}\left(\frac{u^2}{2}\right)\Bigg|_1^{\frac{1}{2}} = -\frac{1}{\pi}\left(\frac{\left(\frac{1}{2}\right)^2}{2} - \frac{1}{2}\right)$

$\displaystyle= -\frac{1}{\pi}\left(\frac{1}{8} - \frac{1}{2}\right) = -\frac{1}{\pi}\left(-\frac{3}{8}\right) = \boxed{\frac{3}{8\pi}}$

Here $\boxed{\begin{aligned} u &= \sin\left(\frac{\pi}{x}\right) \\ du &= \cos\left(\frac{\pi}{x}\right)\left(-\frac{\pi}{x^2}\right)dx \\ -\frac{1}{\pi}du &= \cos\left(\frac{\pi}{x}\right)\left(\frac{1}{x^2}\right)dx \end{aligned}}$ and $\boxed{\begin{aligned} x = 2 &\implies u = \sin\left(\frac{\pi}{2}\right) = 1 \\ x = 6 &\implies u = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \end{aligned}}$

(g) $\displaystyle\int_{\ln 2}^{\ln 3} \frac{1}{e^{2x}(1-e^{-2x})^2}\,dx = \frac{1}{2}\int_{\frac{3}{4}}^{\frac{8}{9}} \frac{1}{u^2}\,du = -\frac{1}{2u}\Bigg|_{\frac{3}{4}}^{\frac{8}{9}} = -\frac{1}{2}\left(\frac{1}{\frac{8}{9}} - \frac{1}{\frac{3}{4}}\right)$

$\displaystyle= -\frac{1}{2}\left(\frac{9}{8} - \frac{4}{3}\right) = -\frac{1}{2}\left(\frac{27}{24} - \frac{32}{24}\right) = -\frac{1}{2}\left(-\frac{5}{24}\right) = \boxed{\frac{5}{48}}$

Here $\boxed{\begin{aligned} u &= 1 - e^{-2x} \\ du &= 2e^{-2x}dx \\ \frac{1}{2}du &= \frac{1}{e^{2x}}dx \end{aligned}}$ and $\boxed{\begin{aligned} x = \ln 2 &\implies u = 1 - e^{-2\ln 2} = 1 - e^{\ln(2^{-2})} = 1 - \frac{1}{4} = \frac{3}{4} \\ x = \ln 3 &\implies u = 1 - e^{-2\ln 3} = 1 - e^{\ln(3^{-2})} = 1 - \frac{1}{9} = \frac{8}{9} \end{aligned}}$

4. [10 Points] Find the equation of the tangent line to the curve

$$y = \sin(\ln(1+x)) \; - \; \ln(1+\sin x) \; - \; \frac{e}{1+\ln(1+x)} \; + \; \sin(e^{3x}-1)$$

at the point where $x = 0$.

First compute the corresponding y-value.

$y(0) = \sin(\ln(1+0)) \; - \; \ln(1+\sin 0) \; - \; \dfrac{e}{1+\ln(1+0)} \; + \; \sin(e^0 - 1)$

$= \sin(0) \; - \ln(1) \; - \dfrac{e}{1} + \sin(0) = 0 - 0 - e + 0 = -e$

Second, compute the derivative

$y' = \dfrac{\cos(\ln(1+x))}{1+x} \; - \; \left(\dfrac{\cos x}{1+\sin x}\right) + \dfrac{e}{(1+\ln(1+x))^2}\left(\dfrac{1}{1+x}\right) \; + \; \cos(e^{3x}-1)e^{3x}(3)$

Next find the specific slope when $x = 0$.

$y'(0) = \dfrac{\cos(\ln(1+0))}{1+0} \; - \; \left(\dfrac{\cos 0}{1+\sin 0}\right) + \dfrac{e}{(1+\ln(1+0))^2}\left(\dfrac{1}{1+0}\right) \; + \; \cos(e^0-1)e^0(3)$

$= \cos(0) \; - \; \left(\dfrac{\cos 0}{1}\right) + \dfrac{e}{(1+0)^2}\left(\dfrac{1}{1+0}\right) \; + \; \cos(0)(1)(3) = 1 - 1 + e + 3 = e + 3$

Finally, use *point-slope form* to get the equation of the tangent line $y - (-e) = (e+3)(x-0)$.

We have $\boxed{y = (e+3)x - e}$

5. [20 Points] Consider the function given by

$$f(x) = e^x + \frac{1}{e^x} + x^e + \frac{1}{x^e} + e^e + \frac{1}{e^e} + \frac{x}{e} + \frac{e}{x} + ex + \frac{1}{ex}$$

(a) Compute the **derivative**, $f'(x)$.

First rewrite

$f(x) = e^x + e^{-x} + x^e + x^{-e} + e^e + \dfrac{1}{e^e} + \dfrac{1}{e}x + ex^{-1} + ex + \dfrac{1}{e}x^{-1}$

Next, $f'(x) = \boxed{e^x - e^{-x} + x^e - ex^{-e-1} + 0 + 0 + \dfrac{1}{e} - ex^{-2} + e - \dfrac{1}{e}x^{-2}}$

(b) Compute the **antiderivative**, $\displaystyle\int f(x)\, dx$.

$\displaystyle\int f(x)\, dx = \int e^x + e^{-x} + x^e + x^{-e} + e^e + \frac{1}{e^e} + \frac{1}{e}x + \frac{e}{x} + ex + \frac{1}{e}\left(\frac{1}{x}\right) \; dx$

$= \boxed{e^x - e^{-x} + \dfrac{x^{e+1}}{e+1} + \dfrac{x^{-e+1}}{-e+1} + e^e x + \dfrac{1}{e^e}x + \dfrac{1}{e}\left(\dfrac{x^2}{2}\right) + e\ln|x| + \dfrac{ex^2}{2} + \dfrac{1}{e}\ln|x| + C}$

6. [15 Points] Compute $\displaystyle\int_1^4 4 - x - x^2\,dx$ using each of the following two different methods:

(a) Fundamental Theorem of Calculus,
(b) The limit definition of the definite integral.

(a) Fundamental Theorem of Calculus

$$\int_1^4 4 - x - x^2\,dx = 4x - \frac{x^2}{2} - \frac{x^3}{3}\Big|_1^4 = \left(16 - 8 - \frac{64}{3}\right) - \left(4 - \frac{1}{2} - \frac{1}{3}\right) = \boxed{-\frac{33}{2}}$$

(b) Limit definition of the definite integral

Here $a = 1, b = 4, \Delta x = \dfrac{b-a}{n} = \dfrac{4-1}{n} = \dfrac{3}{n}$, and $x_i = a + i\Delta x = 1 + \dfrac{3i}{n}$.

$$\int_1^4 4 - x - x^2\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x \;= \lim_{n\to\infty}\sum_{i=1}^{n} f\left(1 + \frac{3i}{n}\right)\frac{3}{n}$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(4 - \left(1 + \frac{3i}{n}\right) - \left(1 + \frac{3i}{n}\right)^2\right)\frac{3}{n}$$

$$= \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n} 4 - 1 - \frac{3i}{n} - \left(1 + \frac{6i}{n} + \frac{9i^2}{n^2}\right)$$

$$= \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n} 2 - \frac{9i}{n} - \frac{9i^2}{n^2}$$

$$= \lim_{n\to\infty}\left(\frac{3}{n}\sum_{i=1}^{n} 2 - \frac{3}{n}\sum_{i=1}^{n}\frac{9i}{n} - \frac{3}{n}\sum_{i=1}^{n}\frac{9i^2}{n^2}\right)$$

$$= \lim_{n\to\infty}\left(\frac{6}{n}\sum_{i=1}^{n} 1 - \frac{27}{n^2}\sum_{i=1}^{n} i - \frac{27}{n^3}\sum_{i=1}^{n} i^2\right)$$

$$= \lim_{n\to\infty}\left(\frac{6}{n}(n) - \frac{27}{n^2}\frac{n(n+1)}{2} - \frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right)$$

$$= \lim_{n\to\infty}\left(6 - \frac{27}{2}\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right) - \frac{27}{6}\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)\right)$$

$$= \lim_{n\to\infty}\left(6 - -\frac{27}{2}(1)\left(1 + \frac{1}{n}\right) - \frac{27}{6}(1)\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)\right)$$

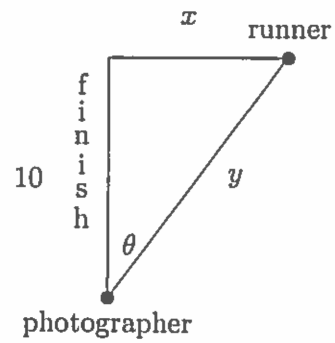
$$= 6 - \frac{27}{2} - \frac{54}{6}$$

$$= 6 - \frac{27}{2} - 9 = \boxed{-\frac{33}{2}} \quad \text{Match}$$

Recall $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ and $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\sum_{i=1}^{n} 1 = n$

7. [10 Points] A photographer is televising a race from a position that is 10 feet from the track in line with the finish line. When the lead runner is 8 feet from the finish line, he is jogging at a rate of 4 feet per second. At this moment, how fast is the photographer rotating to keep the runner in direct view of the camera?

- Diagram



- Variables

Let x = distance from runner to finish line at time t
Let y = distance from photographers to runner at time t
Let θ = angle between camera focus and finish line at time t
Find $\frac{d\theta}{dt}$ =? when $x = 8$ feet
and $\frac{dx}{dt} = -4\frac{\text{ft}}{\text{sec}}$

- Equation relating the variables:

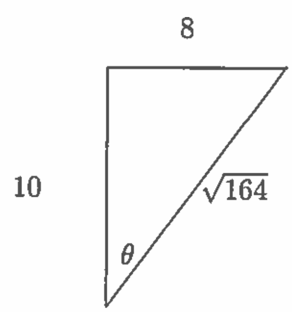
$\tan\theta = \frac{x}{10}$

- Differentiate both sides w.r.t. time t.

$\sec^2\theta\frac{d\theta}{dt} = \frac{1}{10}\frac{dx}{dt}$

- Extra solvable information:

$x^2 + 100 = y^2 \Rightarrow y = \sqrt{164}$



Finally, read $\sec\theta$ off of the triangle.

Here $\sec\theta = \frac{\sqrt{164}}{10}$

- Substitute Key Moment Information (now and not before now!!!):

$$\left(\frac{\sqrt{164}}{10}\right)^2 \frac{d\theta}{dt} = \frac{1}{10}(-4)$$

- Solve for the desired quantity:

$$\frac{d\theta}{dt} = -\frac{4}{10}\left(\frac{100}{164}\right) = \boxed{-\frac{10}{41}\frac{\text{rad}}{\text{sec}}}$$

- Answer the question that was asked: The camera is rotating at a rate of $\frac{10}{41}$ radians every second.

8. [15 Points] Let $f(x) = (x-2)^2 e^x$.

For this function, discuss domain, vertical and horizontal asymptote(s), interval(s) of increase or decrease, local extreme value(s), concavity, and inflection point(s). Then use this information to present a detailed and labelled sketch of the curve.

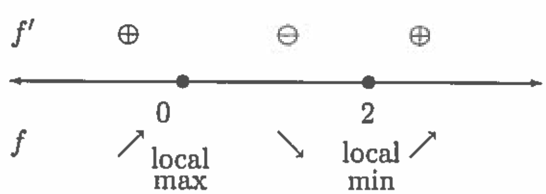
Take my word that $\lim_{x\to\infty} f(x) = +\infty$ and $\lim_{x\to-\infty} f(x) = 0$.

Also take my word that $f'(x) = x(x-2)e^x$ and $f''(x) = (x^2-2)e^x$.

- Domain$=\mathbb{R}$.
- It has no vertical asymptotes.
- There is a horizontal asymptote for this f at $y=0$ because $\lim_{x\to-\infty} f(x) = 0$.
- First Derivative Information

We use the given derivative $f'(x) = x(x-2)e^x$ to find critical numbers. The critical points occur where f' is undefined (never here) or zero. The latter happens when $x=0$ or $x=2$.

Using sign testing/analysis for f',

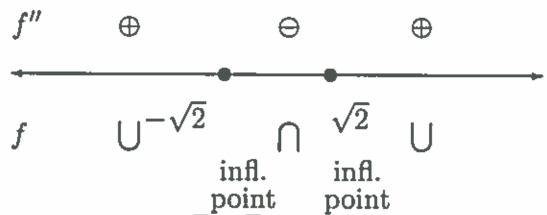


So f is increasing on the intervals $(-\infty, 0)$ and $(2,\infty)$. and f is decreasing on $(0,2)$. Moreover, f has a local max at $x=0$ with $f(0)=4$, and a local min at $x=2$ with $f(2)=0$.

- Second Derivative Information

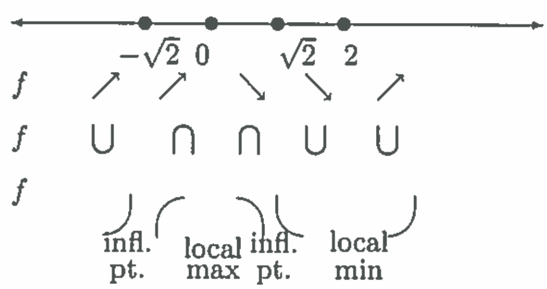
Setting the given second derivative $f'' = (x^2-2)e^x = 0$ we solve for our possible inflection points $x = \pm\sqrt{2}$.

Using sign testing/analysis for f'',



So f is concave down on the interval $(-\sqrt{2}, \sqrt{2})$ and concave up on $(-\infty, -\sqrt{2})$ and $(\sqrt{2}, \infty)$, with inflection points at $(-\sqrt{2}, (-\sqrt{2}-2)^2 e^{-\sqrt{2}})$ and $(\sqrt{2}, (\sqrt{2}-2)^2 e^{\sqrt{2}})$. (yes, not nice values...)

- Piece the first and second derivative information together



Sketch: See Next Page.

9. [20 Points]

(a) Consider the region bounded by $y = e^x + 1$, $y = 4$, and $x = 0$. Sketch the bounded region.

See me for a shaded sketch. It is the area bounded by the three curves below.

Note that the curves intersect when $1 + e^x = 4$ which is when $e^x = 3$ which implies $x = \ln 3$.

Sketch: See Next Page.

(b) Compute the area of the bounded region in (a).

$$\text{Area} = \int_0^{\ln 3} \text{top} - \text{bottom}\ dx = \int_0^{\ln 3} 4 - (e^x + 1)\ dx = \int_0^{\ln 3} 3 - e^x\ dx = 3x - e^x \Big|_0^{\ln 3}$$

$$= (3\ln 3 - e^{\ln 3}) - (0 - e^0) = 3\ln 3 - 3 + 1 = \boxed{3\ln 3 - 2} \overset{\text{or}}{=} \boxed{\ln 27 - 2}$$

(c) Compute the volume of the three-dimensional solid obtained by rotating the region in (a) about the **x-axis**. Sketch the solid, along with one of the approximating washers.

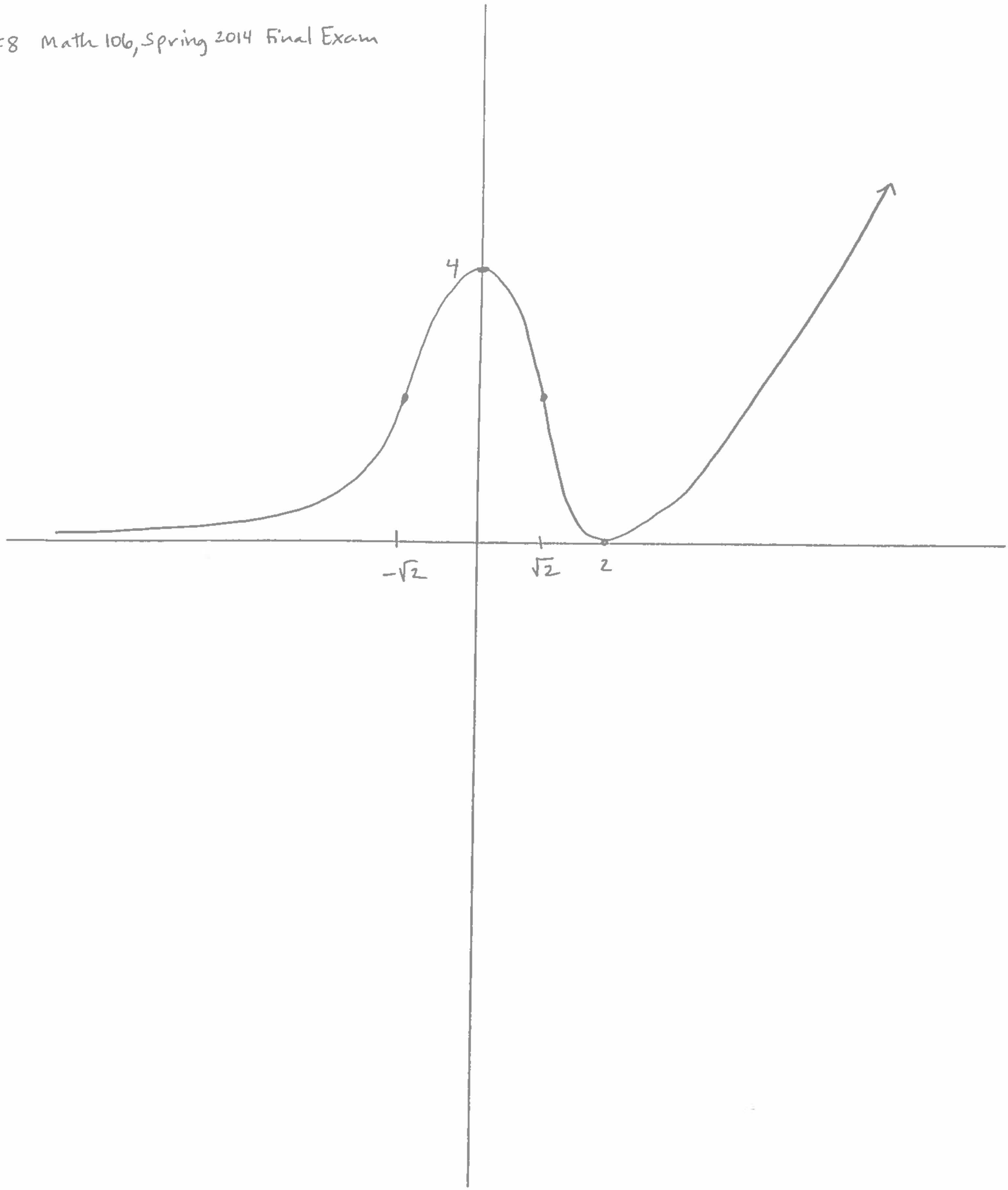
See me for a sketch. See Next Page.

$$\text{Volume} = \int_0^{\ln 3} \pi[(\text{outer radius})^2 - (\text{inner radius})^2]\ dx = \pi \int_0^{\ln 3} 4^2 - (1 + e^x)^2\ dx$$

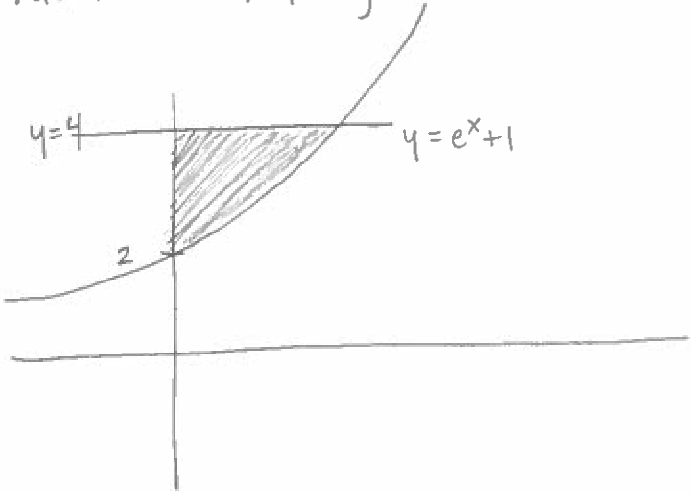
Sketch for

$$f(x) = (x-2)^2 e^x$$

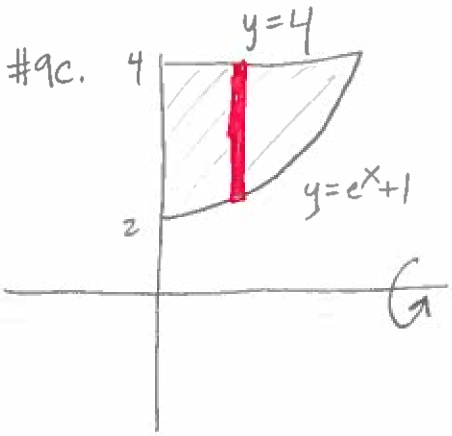
#8 Math 106, Spring 2014 Final Exam

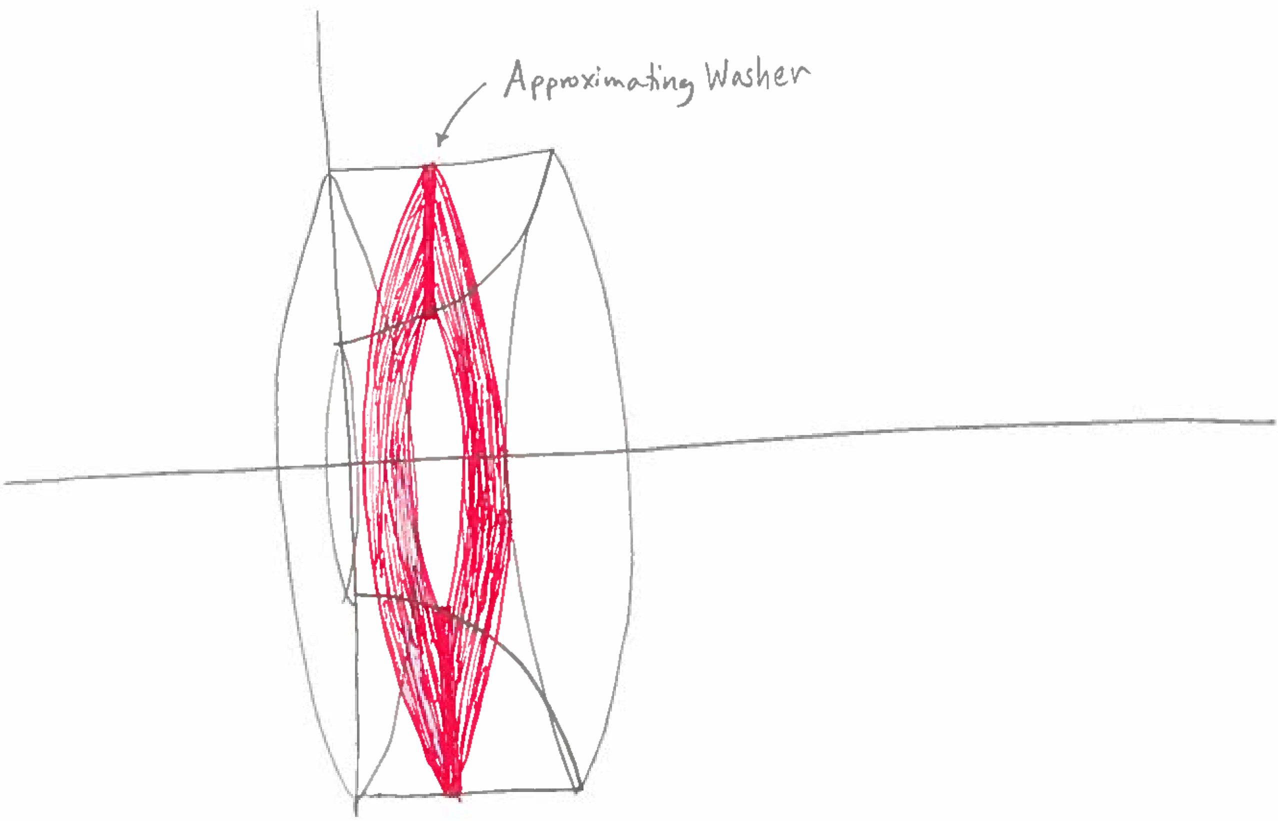


#9a. Math106, Spring 2014 Final Exam



#9c.





$$= \pi \int_0^{\ln 3} 16 - (1 + 2e^x + e^{2x})\ dx = \pi \int_0^{\ln 3} 16 - 1 - 2e^x - e^{2x}\ dx = \pi \int_0^{\ln 3} 15 - 2e^x - e^{2x}\ dx$$

$$= \pi[15x - 2e^x - \tfrac{1}{2}e^{2x}]\Big|_0^{\ln 3} = \pi[(15\ln 3 - 2e^{\ln 3} - \tfrac{1}{2}e^{2\ln 3}) - (0 - 2e^0 - \tfrac{1}{2}e^0)]$$

$$= \pi[15\ln 3 - 6 - \tfrac{1}{2}e^{\ln(3^2)} + 2 + \tfrac{1}{2}]$$

$$= \pi[15\ln 3 - 6 - \tfrac{9}{2} + 2 + \tfrac{1}{2}] = \pi[15\ln 3 - 4 - \tfrac{8}{2}] = \pi[15\ln 3 - 4 - 4] = \boxed{\pi[15\ln 3 - 8]}$$

(d) Consider a **different** region bounded by $y = \sin x$, $y = 1$, $x = 0$ and $x = \frac{\pi}{2}$. Sketch the bounded region.

See me for a shaded sketch. It is the area bounded by the four curves below.

Sketch: See Next Page.

(e) **Set-Up** but **DO NOT EVALUATE** the integral to compute the volume of the three-dimensional solid obtained by rotating the region in (d) about the horizontal line $y = -1$. Sketch the solid, along with one of the approximating washers.

See me for a sketch. See Next Page.

Volume= $\int_0^{\frac{\pi}{2}} \pi[(\text{outer radius})^2 - (\text{inner radius})^2]\ dx = \boxed{\pi \int_0^{\frac{\pi}{2}} 2^2 - (1 + \sin x)^2\ dx}$

(f) Consider a **different** region bounded by $y = \ln x$, $y = 0$, $y = 3$ and $x = 0$. Sketch the bounded region.

See me for a shaded sketch. It is the area bounded by the four curves below.

(g) **Set-Up** but **DO NOT EVALUATE** the integral to compute the volume of the three-dimensional solid obtained by rotating the region in (f) about the **y-axis**. Sketch the solid, along with one of the approximating disks.

See me for a sketch. See Next Page.

Note that $y = \ln x \Rightarrow x = e^y$.

Volume= $\int_0^3 \pi(\text{ radius})^2\ dy = \pi \int_0^3 (e^y)^2\ dy = \boxed{\pi \int_0^3 e^{2y}\ dy}$

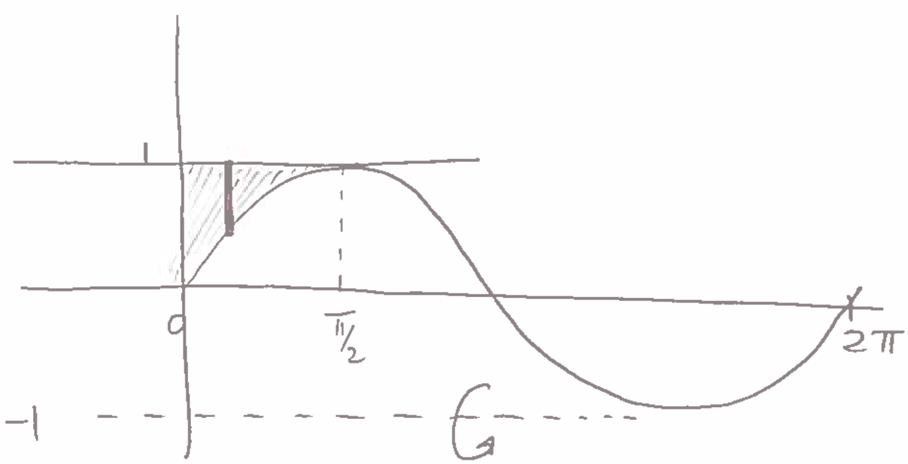
10. [15 Points] Consider an object moving on a number line such that its velocity at time t seconds is given by $\boldsymbol{v(t) = t^2 - 2t}$ feet per second. Also assume that the position of the object at one second is three feet.

(a) Compute the acceleration function $a(t)$ and the position function $s(t)$.

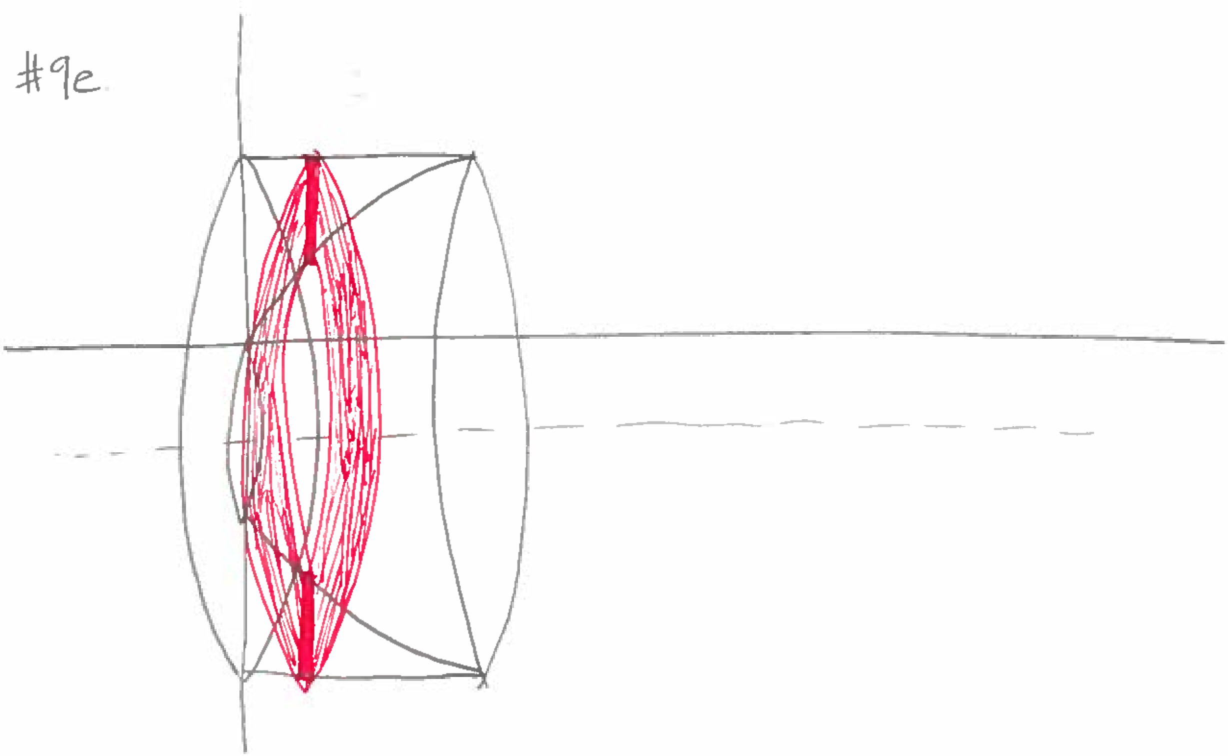
$a(t) = v'(t) = \boxed{2t - 2}$ ft/sec

$s(t) = \int v(t)\ dt = \int t^2 - 2t\ dt = \frac{t^3}{3} - t^2 + C$

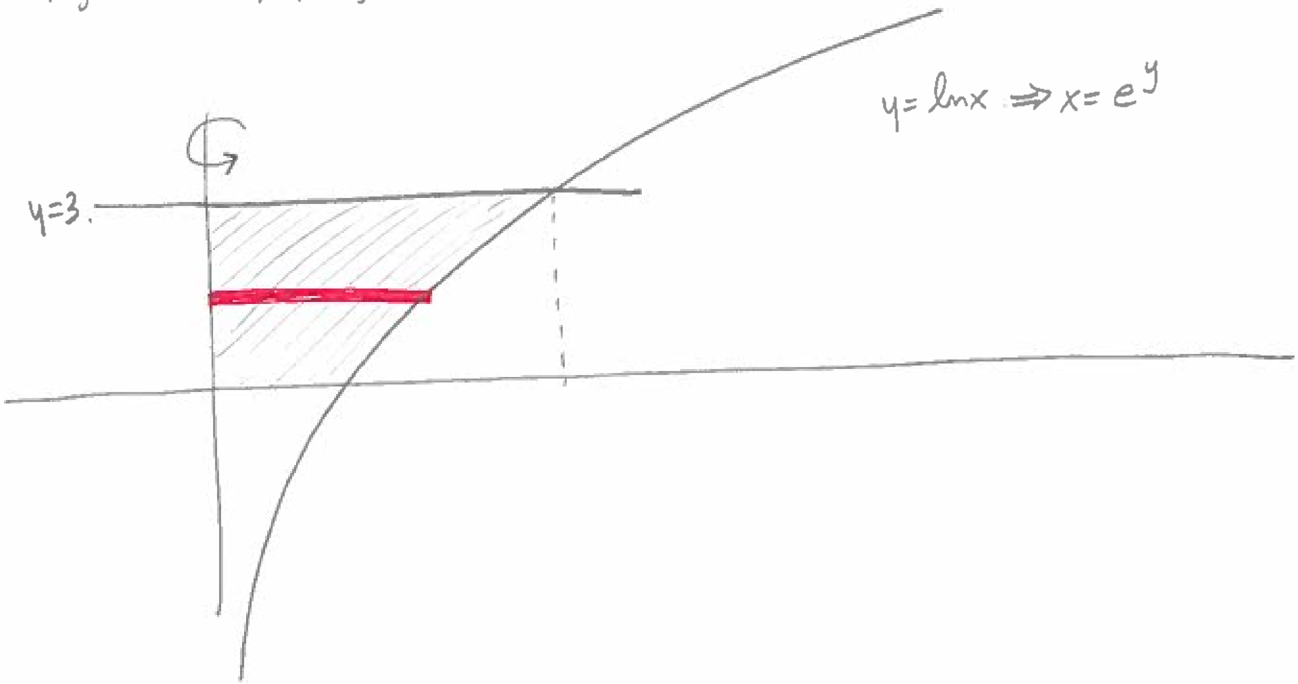
#9d. Math 106, Spring 2014 Final Exam.

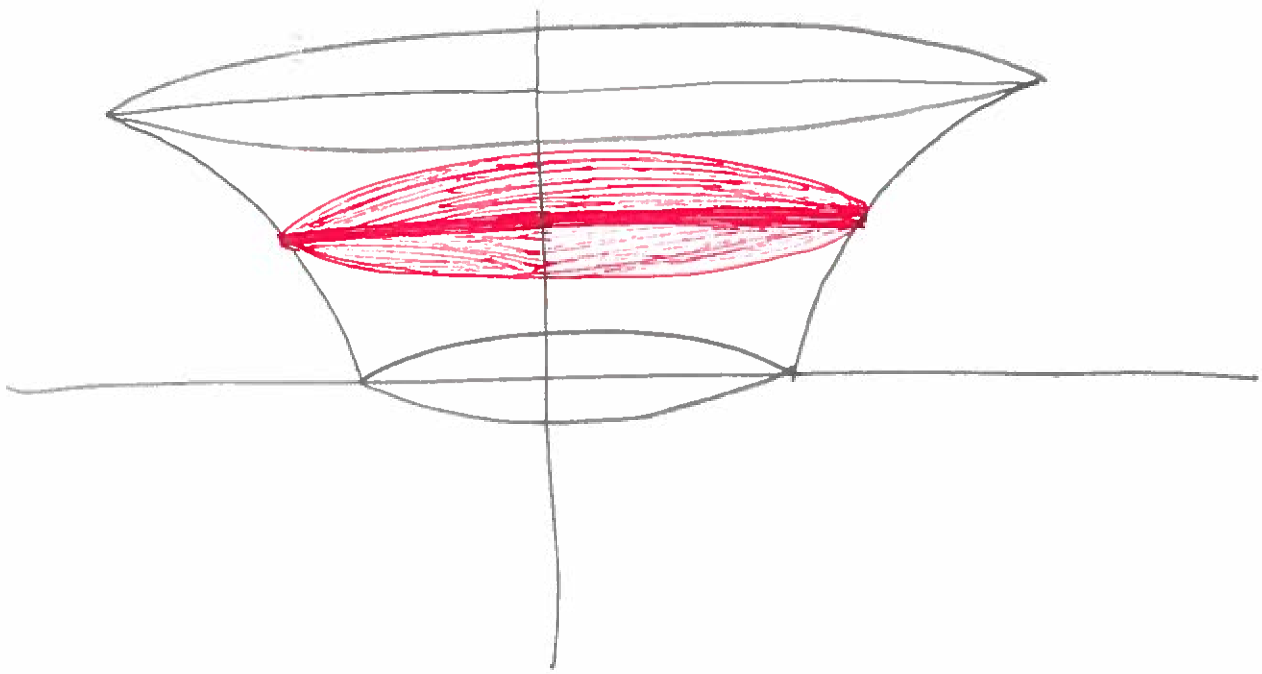


#9e.



#9f, Math 106, Spring 14 Final Exam





Use the initial condition $s(1) = 3$

$s(1) = \frac{1}{3} - 1 + C \overset{\text{set}}{=} 3 \Rightarrow C = \frac{11}{3}$

Finally, $s(t) = \boxed{\frac{t^3}{3} - t^2 + \frac{11}{3}}$

(b) Sketch $v(t)$. Next sketch $|v(t)|$.

Sketch for $v(t)$.

Sketch for $|v(t)|$.

See Next Page

(c) Compute the **total distance** travelled for $0 \le t \le 3$.

$$\text{Total Distance} = \int_0^3 |v(t)|\,dt = \int_0^3 |t^2 - 2t|\,dt = \int_0^2 |t^2 - 2t|\,dt + \int_2^3 |t^2 - 2t|\,dt$$

$$= \int_0^2 -(t^2 - 2t)\,dt + \int_2^3 t^2 - 2t\,dt$$

$$= -\frac{t^3}{3} + t^2\Big|_0^2 + \left(\frac{t^3}{3} - t^2\right)\Big|_2^3$$

$$= \left(-\frac{8}{3} + 4\right) - (0 + 0) + (9 - 9) - \left(\frac{8}{3} - 4\right)$$

$$= -\frac{8}{3} + 4 - \frac{8}{3} + 4$$

$$= 8 - \frac{16}{3}$$

$$= \frac{24}{3} - \frac{16}{3}$$

$$= \boxed{\frac{8}{3}}$$

11. [10 Points] A population of bacteria was growing exponentially. Initially there were 2 cells. After 1 hour there were 8 cells. How many cells were there after 4 hours? When were there 128 cells?

The solution is given by $P(t) = P(0)e^{kt}$. We know $P(0) = 2$, $P(1) = 8$, $P(4) = ?$, $P(?) = 128$.

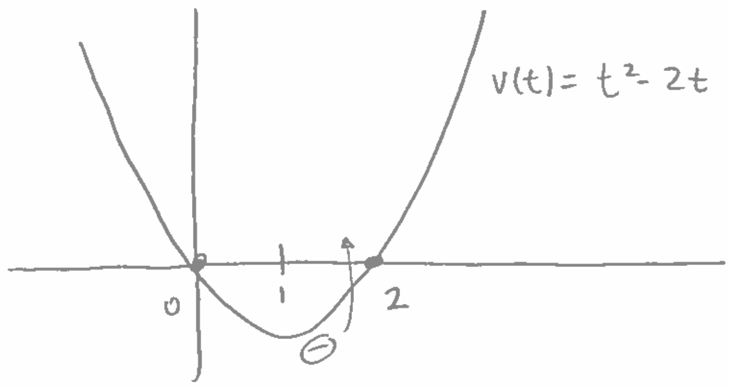
First set $P(0) = 2$. The solution becomes $P(t) = 2e^{kt}$.

Next use the given information $P(1) = 8$ to solve for k.

$P(1) = 2e^{k(1)} \overset{\text{set}}{=} 8$. Then $e^k = 4 \Rightarrow k = \ln 4$.

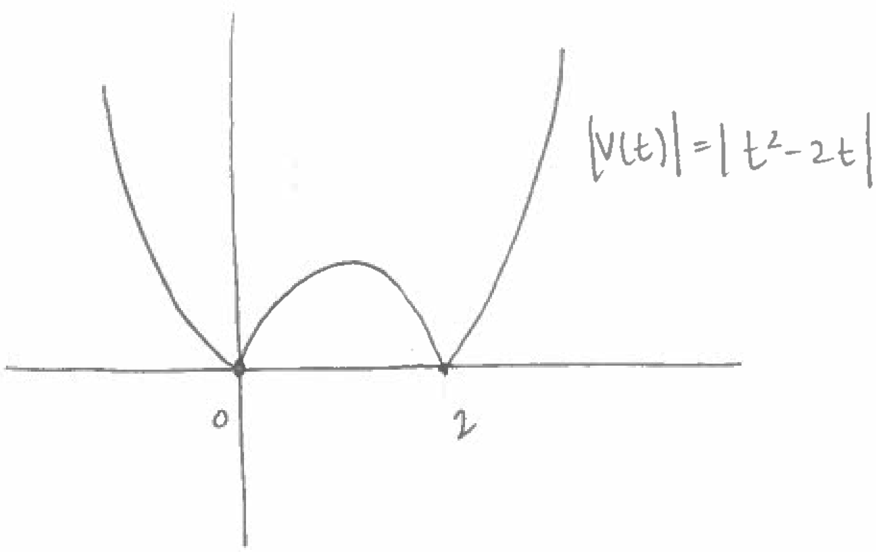
The solution now becomes

#10 b. Math 106, Spring 14 Final Exam



$V(t) = t^2 - 2t = t(t-2) = 0.$

$t = 0 \quad t = 2$ zeroes.



$P(t) = 2e^{(\ln 4)t} = 2e^{t \ln 4} = 2e^{\ln 4^t} = 2 \cdot 4^t.$

Now use this equation to answer the last two questions:

$P(4) = 2 \cdot 4^4 = 2(246) = 512$ cells.

and

$P(t) = 2 \cdot 4^t \overset{\text{set}}{=} 128 \Rightarrow 4^t = 64 \Rightarrow t = 3$ hours.

There are 512 cells after 4 hours, and there were 128 cells after 3 hours.