

Homework #10

Due **Wednesday, March 4th** in Gradescope by 11:59 pm ET**Goal:** Computing Area using the Limit Definition of the Definite Integral with Riemann SumsDefinition: the **Definite Integral** of a function f from $x = a$ to $x = b$ is given by

$$\begin{aligned}
 (\bullet) \quad \int_a^b f(x) \, dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x \\
 &= \lim_{n \rightarrow \infty} [f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \dots + f(x_i)\Delta x + \dots + f(x_n)\Delta x]
 \end{aligned}$$

Note: The Definite Integral is a Limit of a Sum of areas! Just think about this formula as

*the **Limiting Value** of the sum of the areas of finitely many (n) approximating rectangles.*To compute definite integrals the *long (limit) way*, follow these steps:Step 1: Given the integral $\int_a^b f(x) \, dx$, **pick off** or **identify** the **integrand** $f(x)$, and **limits of integration** (lower limit) a and (upper limit) b .Step 2: Compute $\Delta x = \frac{b-a}{n}$. This Width of each partitioned interval will be in terms of n .Step 3: Compute $x_i = a + i\Delta x$. Leave the i as your counter. You have the left-most endpoint a from Step 1. You have width Δx from Step 2. This endpoint x_i should be in terms of i and n .Step 4: Plug x_i and Δx into the formula (\bullet) above. Here it is again:

$$(\bullet) \quad \int_a^b f(x) \, dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x \quad \leftarrow \text{MEMORIZE!}$$

Step 5: Use the following formulas for sum of integers i and finish evaluating the limit in n .

$$\sum_{i=1}^n 1 = n \qquad (*) \quad \sum_{i=1}^n i = \frac{n(n+1)}{2} \qquad (**) \quad \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$(***) \quad \sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2} \right)^2$$

Note: your final answer for the definite integral should be a **number** after you finish the limit.

Read through the entire next problem. Make sure you understand the formula to start, as well as the formulas for Δx and x_i . Because it doesn't feel natural yet, just trust the formulas right now. Use arrows to justify the final limiting size argument.

Evaluate $\int_0^6 x^2 dx$ using the Limit Definition of the Definite Integral (and Riemann Sums).

Here $f(x) = x^2$, $a = 0$, $b = 6$, $\Delta x = \frac{b-a}{n} = \frac{6-0}{n} = \frac{6}{n}$ and $x_i = a + i\Delta x = 0 + i\left(\frac{6}{n}\right) = \frac{6i}{n}$.

$$\begin{aligned}
 \int_0^6 x^2 dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x = \lim_{n \rightarrow \infty} \sum_{i=1}^n f\left(\frac{6i}{n}\right) \left(\frac{6}{n}\right) \\
 &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\left(\frac{6i}{n}\right)^2\right) \frac{6}{n} \\
 &= \lim_{n \rightarrow \infty} \frac{6}{n} \sum_{i=1}^n \frac{36i^2}{n^2} \quad \text{factor all non-}i \text{ pieces out} \\
 &= \lim_{n \rightarrow \infty} \left(\frac{216}{n^3} \sum_{i=1}^n i^2\right) \\
 &= \lim_{n \rightarrow \infty} \left(\frac{216}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}\right) \quad \text{using (**)} \\
 &= \lim_{n \rightarrow \infty} \left(\frac{216}{6} \cdot \frac{n(n+1)(2n+1)}{n^3}\right) \\
 &= \lim_{n \rightarrow \infty} \left(\frac{216}{6} \cdot \frac{n(n+1)(2n+1)}{n \cdot n \cdot n}\right) \quad \text{repartner} \\
 &= \lim_{n \rightarrow \infty} \left(\frac{216}{6} \cdot \left(\frac{n}{n}\right) \cdot \left(\frac{n+1}{n}\right) \cdot \left(\frac{2n+1}{n}\right)\right) \quad \text{split} \\
 &= \lim_{n \rightarrow \infty} \left(\frac{216}{6} \cdot 1 \cdot \left(1 + \frac{1}{n}\right) \cdot \left(2 + \frac{1}{n}\right)\right) \\
 &= \frac{216}{6} \cdot 1 \cdot 2 = \frac{216}{3} = \boxed{72}
 \end{aligned}$$

Read through the entire next problem. Make sure you understand the formula to start, as well as the formulas for Δx and x_i . Here the lower limit of integration a is **not** 0.

Evaluate $\int_1^4 6 - 3x \, dx$ using the Limit Definition of the Definite Integral (and Riemann Sums).

Here $f(x) = 6 - 3x$, $a = 1$, $b = 4$, $\Delta x = \frac{b-a}{n} = \frac{4-1}{n} = \frac{3}{n}$

and $x_i = a + i\Delta x = 1 + i\left(\frac{3}{n}\right) = 1 + \frac{3i}{n}$.

$$\begin{aligned}
 \int_1^4 6 - 3x \, dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i)\Delta x = \lim_{n \rightarrow \infty} \sum_{i=1}^n f\left(1 + \frac{3i}{n}\right) \left(\frac{3}{n}\right) \\
 &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(6 - 3\left(1 + \frac{3i}{n}\right)\right) \frac{3}{n} \\
 &= \lim_{n \rightarrow \infty} \left(\frac{3}{n} \sum_{i=1}^n \left(3 - \frac{9i}{n}\right)\right) \text{ next distribute} \\
 &= \lim_{n \rightarrow \infty} \left(\frac{3}{n} \left(\sum_{i=1}^n 3 - \sum_{i=1}^n \frac{9i}{n}\right)\right) \text{ split sums} \\
 &= \lim_{n \rightarrow \infty} \left(\frac{3}{n} \sum_{i=1}^n 3 - \frac{3}{n} \sum_{i=1}^n \frac{9i}{n}\right) \\
 &= \lim_{n \rightarrow \infty} \left(\frac{9}{n} \sum_{i=1}^n 1 - \frac{27}{n^2} \sum_{i=1}^n i\right) \text{ pull out non-}i \text{ pieces} \\
 &= \lim_{n \rightarrow \infty} \frac{9}{n} (\cancel{n}) - \frac{27}{n^2} \left(\frac{n(n+1)}{2}\right) \text{ using (*)} \\
 &= \lim_{n \rightarrow \infty} 9 - \frac{27}{2} \left(\frac{\cancel{n}}{n}\right) \left(\frac{n+1}{n}\right) \text{ repartner} \\
 &= \lim_{n \rightarrow \infty} 9 - \frac{27}{2} (1) \left(1 + \frac{1}{\cancel{n}}\right) \text{ split} \\
 &= 9 - \frac{27}{2} = \frac{18}{2} - \frac{27}{2} = \boxed{\frac{9}{2}}
 \end{aligned}$$

Read through the entire next problem. The lower limit of integration a is **negative** this time.

Evaluate $\int_{-2}^3 x^2 - 4x + 3 \, dx$ using the Limit Definition of the Definite Integral.

$$\text{Here } f(x) = x^2 - 4x + 3, \quad a = -2, \quad b = 3, \quad \Delta x = \frac{b-a}{n} = \frac{3 - (-2)}{n} = \frac{5}{n}$$

$$\text{and } x_i = a + \Delta x \cdot i = -2 + i \left(\frac{5}{n} \right) = -2 + \frac{5i}{n}.$$

$$\begin{aligned} \int_{-2}^3 x^2 - 4x + 3 \, dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x = \lim_{n \rightarrow \infty} \sum_{i=1}^n f \left(-2 + \frac{5i}{n} \right) \left(\frac{5}{n} \right) \\ &= \lim_{n \rightarrow \infty} \frac{5}{n} \sum_{i=1}^n \left(-2 + \frac{5i}{n} \right)^2 - 4 \left(-2 + \frac{5i}{n} \right) + 3 \\ &= \lim_{n \rightarrow \infty} \frac{5}{n} \sum_{i=1}^n \left(4 - \frac{20i}{n} + \frac{25i^2}{n^2} + 8 - \frac{20i}{n} + 3 \right) \quad \text{FOIL Algebra} \\ &= \lim_{n \rightarrow \infty} \frac{5}{n} \sum_{i=1}^n \left(\frac{25i^2}{n^2} - \frac{40i}{n} + 15 \right) \quad \text{now distribute coeff/sum} \\ &= \lim_{n \rightarrow \infty} \frac{5}{n} \sum_{i=1}^n \frac{25i^2}{n^2} - \frac{5}{n} \sum_{i=1}^n \frac{40i}{n} + \frac{5}{n} \sum_{i=1}^n 15 \\ &= \lim_{n \rightarrow \infty} \frac{125}{n^3} \sum_{i=1}^n i^2 - \frac{200}{n^2} \sum_{i=1}^n i + \frac{75}{n} \sum_{i=1}^n 1 \quad \text{now use } (*) / (**), \text{ with } \nearrow n \\ &= \lim_{n \rightarrow \infty} \frac{125}{n^3} \left(\frac{n(n+1)(2n+1)}{6} \right) - \frac{200}{n^2} \left(\frac{n(n+1)}{2} \right) + \frac{75}{n} (n) \\ &= \lim_{n \rightarrow \infty} \frac{125}{6} \left(\frac{n}{n} \right) \left(\frac{n+1}{n} \right) \left(\frac{2n+1}{n} \right) - \frac{200}{2} \left(\frac{n}{n} \right) \left(\frac{n+1}{n} \right) + 75 \\ &= \lim_{n \rightarrow \infty} \frac{125}{6} (1) \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right) - (100) (1) \left(1 + \frac{1}{n} \right) + 75 \\ &= \frac{125}{6} (1)(1)(2) - (100)(1)(1) + 75 = \frac{125}{3} - 100 + 75 \\ &= \frac{125}{3} - 25 = \frac{125}{3} - \frac{75}{3} = \boxed{\frac{50}{3}} \end{aligned}$$

★ **Now complete these Homework problems:**

1. Compute by hand, manually, the Area bounded above by the graph of $y = 2x + 5$ and bounded below by $y = 0$ and between $x = 0$ and $x = 3$. Sketch the graph and shade the bounded region.

2. Evaluate $\int_0^3 2x + 5 \, dx$ using the Limit Definition of the Definite Integral (and Riemann Sums)

3. Compute by hand, manually, the **Net Area** bounded between the graph of $y = 4 - 2x$ and the x -axis ($y = 0$) and between $x = 1$ and $x = 5$. Sketch the graph and shade the bounded region.

4. Evaluate $\int_1^5 4 - 2x \, dx$ using the Limit Definition of the Definite Integral (and Riemann Sums)

NOTE: Recall the Definite Integral computes the Area bounded above the x -axis *minus* the Area bounded below the x -axis.

5. Evaluate $\int_0^4 x^2 \, dx$ using the Limit Definition of the Definite Integral (and Riemann Sums). Sketch the graph and shade the bounded region.

6. Evaluate $\int_{-1}^2 x^2 - 3x + 2 \, dx$ using the Limit Definition of the Definite Integral (and Riemann Sums). Sketch the graph and shade the bounded region.

REGULAR OFFICE HOURS

Monday: 12:00–3:00 pm

7:30–9:00 pm TA Uchenna, SMUDD 208A

Tuesday: 1:00–4:00 pm

Wednesday: 1:00-3:15 pm

Thursday: none for Professor

4:00–5:30 pm TA Uchenna, SMUDD 208A

Friday: 12:00–2:00 pm

- Spring Break coming soon!