Exam 3 Spring 2026 Answer Key

1(a)

$$\frac{d}{dx}\ln\left(\frac{\sqrt{9-x^3}\cdot e^{\cos x}}{(1-e^x)^7\cdot(\ln x)}\right)$$

First use Log Algebra to simplify

$$=\frac{d}{dx}\ln\left(\sqrt{9-x^3}\cdot e^{\cos x}\right)-\ln\left((1-e^x)^7\cdot\ln x\right)$$

$$=\frac{d}{dx}\ln(9-x^3)^{\frac{1}{2}}+\cancel{\ln}\left(\cancel{e}^{\cos x}\right)-\left(\ln\left((1-e^x)^7\right)+\ln(\ln x)\right)$$

minuses cancel on last term

$$\overset{\text{prep}}{=}\frac{d}{dx}\frac{1}{2}\ln(9-x^3)+\cos x-7\ln(1-e^x)-\ln(\ln x)$$

$$=\boxed{\frac{1}{2}\left(\frac{1}{9-x^3}\right)\cdot(-3x^2)-\sin x-7\left(\frac{1}{1-e^x}\right)\cdot(-e^x)-\frac{1}{\ln x}\cdot\frac{1}{x}}$$

1(b)

$$\cancel{\ln}(6-5x)=4$$

$$\cancel{e}\qquad\qquad e$$

$$6-5x=e^4$$

$$-5x=e^4-6$$

$$x=\boxed{\frac{e^4-6}{-5}}\overset{\text{OR}}{=}\boxed{\frac{6-e^4}{5}}$$

2.

$$y=(\cos x)^{\ln x}$$

Logarithmic Differentiation

$$\ln y=\ln\left((\cos x)^{\ln x}\right)$$

$$\frac{d}{dx}(\ln y)=\frac{d}{dx}\left((\ln x)\cdot\ln(\cos x)\right)$$

Product Rule on the Right

$$\frac{1}{y}\cdot\frac{dy}{dx}=\ln x\cdot\frac{1}{\cos x}\cdot(-\sin x)+\ln(\cos x)\cdot\frac{1}{x}$$

$$\frac{dy}{dx}=\overset{(\cos x)^{\ln x}}{\cancel{y}}\left(-\frac{\ln x\cdot\sin x}{\cos x}+\frac{\ln(\cos x)}{x}\right)$$

$$=\boxed{(\cos x)^{\ln x}\left(-\frac{\ln x\cdot\sin x}{\cos x}+\frac{\ln(\cos x)}{x}\right)}$$

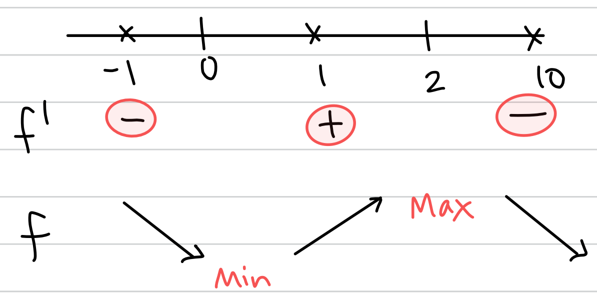
3. $f(x) = \dfrac{x^2}{e^x}$

$$f'(x) = \frac{e^x(2x) - x^2 \cdot e^x}{(e^x)^2} = \frac{\cancel{e^x}\, x(2-x)}{e^{\cancel{2x}}} = \frac{x(2-x)}{\underset{+}{e^x}} \overset{\text{set}}{=} 0$$

$\Rightarrow x = 0$ or $2 - x = 0$

$x = 2$ Critical Numbers

Sign Testing into First Derivative



$$f'(10) = \frac{\overset{+}{10} \cdot \overset{-}{(2-10)}}{\underset{+}{e^{10}}} < 0 \quad (-)$$

$$f'(1) = \frac{\overset{+}{1} \cdot \overset{+}{(2-1)}}{\underset{+}{e^{1}}} > 0 \quad (+)$$

$$f'(-1) = \frac{\overset{-}{(-1)} \cdot \overset{+}{(2-(-1))}}{\underset{+}{e^{-1}}} < 0 \quad (-)$$

$f(0) = \dfrac{0}{e^0} = \dfrac{0}{1} = 0$ Local Minimum Value at $x = 0$

$f(2) = \dfrac{2^2}{e^4} = \dfrac{4}{e^4}$ Local Maximum Value at $x = 2$

4(a) $f(x) = \cos(e^{5x} - 1) + \overset{\text{constant } 0}{\cancel{\ln 1}} + \sin(\ln(1 + 6x))$

$$f'(x) = -\sin(e^{5x} - 1) \cdot 5e^{5x} + 0 + \cos(\ln(1+6x)) \cdot \frac{1}{1+6x} \cdot 6$$

$$f'(0) = -\overset{0}{\cancel{\sin}}(\overset{1}{\cancel{e^0}} - 1) \cdot 5e^0 + \overset{1}{\cancel{\cos}}(\overset{0}{\cancel{\ln}}(1+0)) \cdot \frac{1}{1+0} \cdot 6$$

$= 0 + 6 = 6$ Match!

4(b) $f(x) = e^{\sin(3x)} - \overset{\text{Constant } -7}{\cancel{\ln(e^7)}} - \ln(1 + \sin(5x))$

$$f'(x) = e^{\sin(3x)} \cdot \cos(3x) \cdot 3 - 0 - \frac{1}{1 + \sin(5x)} \cdot \cos(5x) \cdot 5$$

$$f'(0) = \overset{1}{\cancel{e^{\sin 0}}} \cdot \overset{1}{\cancel{\cos 0}} \cdot 3 - \overset{1}{\cancel{\frac{1}{1 + \sin 0}}} \cdot \overset{1}{\cancel{\cos 0}} \cdot 5$$

$= 3 - 5 = -2$ Match!

$$5(a)\quad f(x) = 8\ln x - \frac{8}{e^{8x}} + \overset{\text{constant}}{\cancel{e^{\ln 8}}}^{\,8} - \frac{8}{x} + \overset{\text{constant}}{\ln 8} + \frac{e^{x}}{e^{8x}}$$

simplify

$$= 8\ln x - 8e^{-8x} + 8 - 8x^{-1} + \ln 8 + e^{-7x}$$

$$f'(x) = 8\left(\frac{1}{x}\right) - 8e^{-8x}(-8) + 0 + 8x^{-2} + 0 + e^{-7x}(-7)$$

$$= \boxed{\frac{8}{x} + \frac{64}{e^{8x}} + \frac{8}{x^2} - \frac{7}{e^{7x}}}$$

$$5(b)\quad g(x) = 8e^{x} + \frac{1}{e^{x}} - \cancel{\ln(e^{x})}^{\,x} - \frac{1}{x^8} - \frac{1}{x} + \left(e^{8x}\cdot e^{x}\right)$$

simplify

$$= 8e^{x} + e^{-x} - x - x^{-8} - \frac{1}{x} + e^{9x}$$

$$\int g(x)\,dx = 8e^{x} + \frac{e^{-x}}{-1} - \frac{x^2}{2} \cancel{-} \frac{x^{-7}}{\cancel{-}7} - \ln|x| + \frac{e^{9x}}{9} + C$$

$$= \boxed{8e^{x} - \frac{1}{e^{x}} - \frac{x^2}{2} + \frac{1}{7x^7} - \ln|x| + \frac{e^{9x}}{9} + C}$$

$$6(a)\quad \int \underset{\text{Algebra}}{\frac{6-x^7}{x^8}}\,dx = \int \underset{\text{split}}{\frac{6}{x^8} - \frac{x^7}{x^8}}\,dx \overset{\text{prep}}{=} \int 6x^{-8} - \frac{1}{x}\,dx = \frac{6x^{-7}}{-7} - \ln|x| + C = \boxed{\frac{-6}{7x^7} - \ln|x| + C}$$

$$6(b)\quad \int \frac{x^7}{6-x^8}\,dx = -\frac{1}{8}\int \frac{1}{u}\,du = -\frac{1}{8}\ln|u| + C = \boxed{-\frac{1}{8}\ln|6-x^8| + C}$$

u-sub

$$\boxed{\begin{aligned} u &= 6-x^8 \\ du &= -8x^7\,dx \\ -\tfrac{1}{8}\,du &= x^7\,dx \end{aligned}}$$

$$6(c)\ \int \frac{(1+e^{3x})^2}{e^{3x}}\,dx \overset{\text{FOIL}}{=} \int \frac{1+2e^{3x}+e^{6x}}{e^{3x}}\,dx \underset{\text{Split}}{=} \int \frac{1}{e^{3x}} + \frac{2\cancel{e^{3x}}}{\cancel{e^{3x}}} + \frac{e^{6x}}{e^{3x}}\,dx$$

Algebra

$$= \int e^{-3x} + 2 + e^{3x}\,dx = \frac{e^{-3x}}{-3} + 2x + \frac{e^{3x}}{3} + C$$

$$= \boxed{-\frac{1}{3e^{3x}} + 2x + \frac{e^{3x}}{3} + C}$$

☆ K-rule: $\int e^{kx}dx = \frac{e^{kx}}{k} + C \quad k \neq 0$ constant

$$6(d)\ \int \frac{e^{3x}}{(1+e^{3x})^2}\,dx = \frac{1}{3}\int \frac{1}{u^2}\,du = \frac{1}{3}\left(\frac{u^{-1}}{-1}\right) + C = \frac{-1}{3u} + C = \boxed{\frac{-1}{3(1+e^{3x})} + C}$$

u^{-2}

u-sub

$$\boxed{\begin{aligned} u &= 1+e^{3x} \\ du &= 3e^{3x}dx \\ \tfrac{1}{3}du &= e^{3x}dx \end{aligned}}$$

$$7(a)\ \int_{\pi/6}^{\pi/3} \tan x\,dx = \int_{\pi/6}^{\pi/3} \frac{\sin x}{\cos x}\,dx = -\int_{\sqrt{3}/2}^{1/2} \frac{1}{u}\,du = -\ln|u|\Big|_{\sqrt{3}/2}^{1/2}$$

$$\boxed{\begin{aligned} u &= \cos x \\ du &= -\sin x\,dx \\ -du &= \sin x\,dx \end{aligned}}$$

$$= -\left(\ln\left|\frac{1}{2}\right| - \ln\left|\frac{\sqrt{3}}{2}\right|\right) = \ln\left(\frac{\sqrt{3}}{2}\right) - \ln\left(\frac{1}{2}\right)$$

$$= \ln\left(\frac{\frac{\sqrt{3}}{\cancel{2}}}{\frac{1}{\cancel{2}}}\right) = \ln\sqrt{3} = \ln\left(3^{1/2}\right) = \frac{1}{2}\ln 3$$

Match!

$$\boxed{\begin{aligned} x &= \tfrac{\pi}{6} \rightarrow u = \cos\tfrac{\pi}{6} = \tfrac{\sqrt{3}}{2} \\ x &= \tfrac{\pi}{3} \rightarrow u = \cos\tfrac{\pi}{3} = \tfrac{1}{2} \end{aligned}}$$

7(b) $\displaystyle\int_{e^2}^{e^7} \frac{8}{x\sqrt{2+\ln x}}\,dx = 8\int_4^9 \frac{1}{\sqrt{u}}\,du = 8\cdot\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\Bigg|_4^9 = 16\sqrt{u}\,\Bigg|_4^9$

$u^{-\frac{1}{2}}$

$$\boxed{\begin{aligned} u &= 2+\ln x \\ du &= \frac{1}{x}\,dx \end{aligned}}$$

$$\boxed{\begin{aligned} x = e^2 &\Rightarrow u = 2+\cancel{\ln e}^{2} = 4 \\ x = e^7 &\Rightarrow u = 2+\cancel{\ln e}^{7} = 9 \end{aligned}}$$

$= 16\left(\cancel{\sqrt{9}}^{\,3} - \cancel{\sqrt{4}}^{\,2}\right) = 16$ Match

7(c) $\displaystyle\int_0^{\ln 3} \frac{1}{e^{2x}(1+e^{-2x})^2}\,dx = -\frac{1}{2}\int_2^{\frac{10}{9}} \frac{1}{u^2}\,du = \cancel{-}\frac{1}{2}\left(\frac{u^{-1}}{\cancel{-}1}\right)\Bigg|_2^{\frac{10}{9}}$

u^{-2}

$$\boxed{\begin{aligned} u &= 1+e^{-2x} \\ du &= -2e^{-2x}\,dx \\ -\frac{1}{2}\,du &= \frac{1}{e^{2x}}\,dx \end{aligned}}$$

$= \dfrac{1}{2u}\Bigg|_2^{\frac{10}{9}} = \dfrac{1}{2}\left(\dfrac{1}{\frac{10}{9}} - \dfrac{1}{2}\right)$

$= \dfrac{1}{2}\left(\dfrac{9}{10} - \dfrac{1}{2}\cdot\dfrac{5}{5}\right) = \dfrac{1}{2}\left(\dfrac{9}{10} - \dfrac{5}{10}\right) = \dfrac{1}{\cancel{2}}\left(\dfrac{\cancel{4}^{\,2}}{10}\right) = \dfrac{2}{10} = \dfrac{1}{5}$ Match

$$\boxed{\begin{aligned} x = 0 &\Rightarrow u = 1+e^0 = 2 \\ x = \ln 3 &\Rightarrow u = 1+e^{-2\ln 3} = 1+\cancel{e}^{\ln(3^{-2})} = 1+\frac{1}{9} = \frac{10}{9} \end{aligned}}$$

Bonus #1

$$\lim_{n\to\infty} \frac{2\left(e^{3+\frac{2}{n}} + e^{3+\frac{4}{n}} + e^{3+\frac{6}{n}} + \dots\ e^{3+\frac{2(n-1)}{n}} + e^5\right)}{n}$$

Reverse recognition of Limit Definition of Definite Integral

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{e^{3+\frac{2i}{n}}}_{f(x_i)} \cdot \underbrace{\frac{2}{n}}_{\Delta x} = \int_3^5 e^x\,dx = e^x\Bigg|_3^5$$

$= e^5 - e^3$

$\Delta x = \dfrac{2}{n} = \dfrac{b-a}{n}$

$x_i = a + i\Delta x$

$\quad = 3 + i\cdot\dfrac{2}{n}$

Equivalent to $\displaystyle\int_2^4 e^{x+1}\,dx$ or $\displaystyle\int_1^3 e^{x+2}\,dx$ or $\displaystyle\int_0^2 e^{x+3}$

Bonus #2

$$\int \frac{1}{1-\sin x}\,dx = \int \frac{1}{1-\sin x}\cdot\frac{1+\sin x}{1+\sin x} = \int \frac{1+\sin x}{\underbrace{1-\sin^2 x}_{\cos^2 x}}\,dx$$

$$= \int \frac{1+\sin x}{\cos^2 x}\,dx = \int \overbrace{\frac{1}{\cos^2 x}}^{\sec^2 x} + \frac{\sin x}{\cos^2 x}\,dx$$

$$u = \cos x$$
$$du = -\sin x\,dx$$
$$-du = \sin x\,dx$$

$$= \int \sec^2 x\,dx + \int \frac{\sin x}{(\cos x)^2}\,dx$$

$$= \tan x - \int \overbrace{\frac{1}{u^2}}^{u^{-2}}\,du$$

$$= \tan x \cancel{-} \left(\frac{u^{-1}}{\cancel{-}1}\right) + C$$

$$= \tan x + \frac{1}{u} + C$$

$$= \boxed{\tan x + \frac{1}{\cos x} + C}$$

OR

$$= \boxed{\tan x + \sec x + C}$$