Exam 3 Spring 2025 Answer Key

1. $$y = \ln\left(\frac{(9-x^5)^7\cdot e^{\cos x}}{\sqrt{8-x^3}\cdot\ln(\ln x)}\right) \overset{\text{Simplify prep}}{=} \ln\left((9-x^5)^7\cdot e^{\cos x}\right) - \ln\left(\sqrt{8-x^3}\cdot\ln(\ln x)\right)$$

$$= \ln\left((9-x^5)^{\boxed{7}}\right) + \cancel{\ln e}^{\boxed{\cos x}} - \left(\ln\left((8-x^3)^{\boxed{\frac{1}{2}}}\right) + \ln(\ln(\ln x))\right)$$

$$= 7\ln(9-x^5) + \cos x - \frac{1}{2}\ln(8-x^3) - \ln(\ln(\ln x))$$

$$y' = \boxed{7\cdot\left(\frac{1}{9-x^5}\right)\cdot(-5x^4) - \sin x - \frac{1}{2}\left(\frac{1}{8-x^3}\right)(-3x^2) - \frac{1}{\ln(\ln x)}\cdot\frac{1}{\ln x}\cdot\frac{1}{x}}$$

2. $$y = (\cos x)^{\sqrt{x}}$$ Logarithmic Differentiation

$$\ln y = \ln\left((\cos x)^{\boxed{\sqrt{x}}}\right)$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left(\sqrt{x}\,\ln(\cos x)\right)$$ Product Rule on the Right

$$\frac{1}{\boxed{y}}\cdot\frac{dy}{dx} = \sqrt{x}\cdot\frac{1}{\cos x}\cdot(-\sin x) + \ln(\cos x)\cdot\frac{1}{2\sqrt{x}}$$

$$\frac{dy}{dx} = \overset{(\cos x)^{\sqrt{x}}}{\cancel{y}}\left(-\frac{\sqrt{x}\sin x}{\cos x} + \frac{\ln(\cos x)}{2\sqrt{x}}\right)$$

$$= \boxed{(\cos x)^{\sqrt{x}}\left(-\frac{\sqrt{x}\sin x}{\cos x} + \frac{\ln(\cos x)}{2\sqrt{x}}\right)}$$

3(a) $$f(x) = \sin(\ln(1+7x)) - \ln(1+\cos(4x))$$

$$f'(x) = \cos(\ln(1+7x))\cdot\frac{1}{1+7x}\cdot 7 - \frac{1}{1+\cos(4x)}\cdot(-\sin(4x))\cdot 4$$

$$f'(0) = \overset{1}{\cancel{\cos}}(\overset{0}{\cancel{\ln 1}})\cdot\frac{1}{1+0}\cdot\boxed{7} - \overset{\frac{1}{2}}{\cancel{\frac{1}{1+\underset{1}{\cos 0}}}}\cdot(-\overset{\boxed{0}}{\cancel{\sin 0}})\cdot 4 \xrightarrow{} 0$$

$$= 7 \mp 0 = \boxed{7}$$

3(b) $$f(x) = \frac{e^{8x}}{\underset{e^{5x}}{e^{3x}}} - e^{\sin(6x)} - \cos(1-e^{8x})$$

$$f'(x) = 5e^{5x} - e^{\sin(6x)}\cdot\cos(6x)\cdot 6 \mp \sin(1-e^{8x})\cdot(-e^{8x})\,8$$

$$f'(0) = \boxed{5}\overset{1}{\cancel{e^0}} - \overset{1}{\cancel{e^{\overset{0}{\cancel{\sin 0}}}}}\cdot\overset{1}{\cancel{\cos 0}}\cdot\boxed{6} + \overset{\boxed{0}}{\cancel{\sin}}(\overset{0}{\cancel{1-\overset{1}{\cancel{e^0}}}})(-\overset{1}{\cancel{e^0}})\cdot 8 \xrightarrow{} 0$$

$$= 5 - 6 + 0 = \boxed{-1}$$

4. $f(x) = x^2 \cdot e^x$

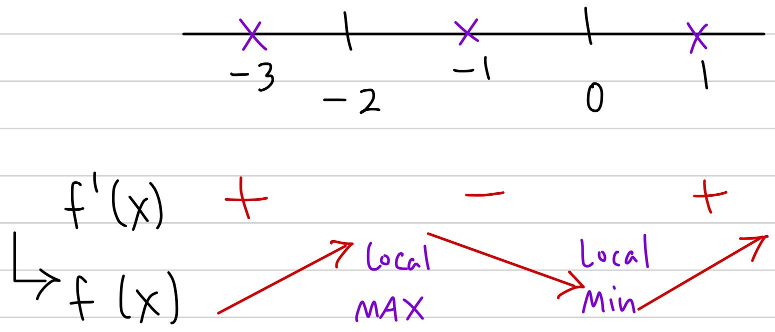
$$f'(x) = x^2 \cdot e^x + e^x(2x) = e^x(x^2+2x) = \overset{\oplus}{e^x} \cdot x(x+2) \overset{\text{set}}{=} 0$$

$e^x \neq 0$

$\Rightarrow x=0$ or $x+2=0$

$\hookrightarrow x=-2$ Critical Numbers

Sign Testing into $f'(x)$



Note e^x Always ⊕

$$f'(-3) = \overset{+}{e^{-3}}\overset{-}{(-3)}\overset{-}{(-3+2)} = +$$

$$f'(-1) = \overset{+}{e^{-1}}\overset{-}{(-1)}\overset{+}{(-1+2)} = -$$

$$f'(1) = \overset{+}{e^{1}}\overset{+}{(1)}\overset{+}{(1+2)} = +$$

☆ $f(x)$ has local Maximum Value of $f(-2) = 4e^{-2} = \boxed{\frac{4}{e^2}}$ occuring when $x=-2$

$f(x)$ has Local Minimum Value of $f(0) = \boxed{0}$ occuring when $x = 0$

$$5(a)\quad \int\left(e^{5x}+\overset{e^{-2x}}{\frac{1}{e^{2x}}}\right)\left(e^{x}-\overset{e^{-7x}}{\frac{1}{e^{7x}}}\right)dx \overset{\text{FOIL}}{=} \int e^{6x}-e^{-2x}+e^{-x}-e^{-9x}\,dx$$

$$= \frac{e^{6x}}{6} \cancel{-} \frac{e^{-2x}}{\cancel{-}2} + \frac{e^{-x}}{-1} \cancel{-} \frac{e^{-9x}}{\cancel{-}9} + C$$

$$= \boxed{\frac{e^{6x}}{6} + \frac{1}{2e^{2x}} - \frac{1}{e^{x}} + \frac{1}{9e^{9x}} + C}$$

K-rules

$\int e^{kx}\,dx = \frac{e^{kx}}{k} + C$ k= constant

$$5(b)\quad \int \frac{7x}{x^2+1}\,dx = 7\cdot\frac{1}{2}\int\frac{1}{u}\,du = \frac{7}{2}\ln|u| + C = \boxed{\frac{7}{2}\ln|x^2+1| + C}$$

$u = x^2+1$
$du = 2x\,dx$
$\frac{1}{2}du = x\,dx$

$$6(a)\quad \int_0^{\frac{\pi}{3}} \tan x\,dx = \int_0^{\frac{\pi}{3}} \frac{\sin x}{\cos x}\,dx = -\int_1^{\frac{1}{2}} \frac{1}{u}\,du = -\ln|u|\Big|_1^{\frac{1}{2}} = -\left(\ln\left(\tfrac{1}{2}\right) - \overset{0}{\cancel{\ln 1}}\right)$$

$$= -\ln\left(\tfrac{1}{2}\right) = -\left(\overset{0}{\cancel{\ln 1}} - \ln 2\right)$$

$$= \cancel{-}(\cancel{-}\ln 2) = \boxed{\ln 2}$$ Match

$u = \cos x$
$du = -\sin x\,dx$
$-du = \sin x\,dx$

$x = 0 \Rightarrow u = \cos 0 = 1$
$x = \frac{\pi}{3} \Rightarrow u = \cos\frac{\pi}{3} = \frac{1}{2}$

6(b) $$\int_{e^2}^{e^9}\frac{1}{x\sqrt{7+\ln x}}\,dx=\int_9^{16}\frac{1}{\sqrt{u}}\,dx=\frac{u^{\frac12}}{\frac12}\Bigg|_9^{16}=2\sqrt{u}\Big|_9^{16}$$

$u^{-\frac12}$

$u=7+\ln x$

$du=\frac{1}{x}\,dx$

$$=2\left(\cancel{\sqrt{16}}^{\,4}-\cancel{\sqrt{9}}^{\,3}\right)=2(4-3)=\boxed{2}$$ Match!

$x=e^2\Rightarrow u=7+\cancel{\ln e}^{2}=7+2=9$

$x=e^9\Rightarrow u=7+\cancel{\ln e}^{9}\ 7+9=16$

6(c) $$\int_0^{\ln 3}\frac{1}{e^{2x}\left(1+e^{-2x}\right)^2}\,dx=-\frac12\int_2^{\frac{10}{9}}\frac{1}{u^2}\,du=\cancel{-}\frac12\left(\frac{u^{-1}}{\cancel{-}1}\right)\Bigg|_2^{\frac{10}{9}}$$

u^{-2}

$u=1+e^{-2x}$

$du=-2e^{-2x}\,dx$

$-\frac12\,du=\frac{1}{e^{2x}}\,dx$

$$=\frac{1}{2u}\Bigg|_2^{\frac{10}{9}}=\frac12\left(\frac{1}{\frac{10}{9}}-\frac12\right)$$

$$=\frac12\left(\frac{9}{10}-\frac12\cdot\frac55\right)=\frac12\left(\frac{9}{10}-\frac{5}{10}\right)=\frac{1}{\cancel{2}}\left(\frac{\cancel{4}^{\,2}}{10}\right)=\frac{2}{10}=\boxed{\frac15}$$ Match

$x=0\Rightarrow u=1+e^0=2$

$x=\ln 3\Rightarrow u=1+e^{-2\ln 3}=1+\cancel{e}^{\cancel{\ln}(3^{-2})}=1+\frac19=\frac{10}{9}$

Optional Bonus

$$\int\frac{(\ln(\ln x))^2}{x\cdot\ln x\,[5+\ln(\ln x)]^2}\,dx=\int\frac{(u-5)^2}{u^2}\,du=\int\frac{u^2-10u+25}{u^2}\,du$$ split-split...

"reverse"

$u=5+\ln(\ln x)\Rightarrow\ln(\ln x)=u-5$

$du=\frac{1}{\ln x}\cdot\frac{1}{x}\,dx$

$25u^{-2}$

$$=\int 1-\frac{10}{u}+\frac{25}{u^2}\,du$$

$$=u-10\ln|u|+25\left(\frac{u^{-1}}{-1}\right)+C$$

$$=\boxed{5+\ln(\ln x)-10\ln|5+\ln(\ln x)-\frac{25}{5+\ln(\ln x)}+C}$$