Exam 3 Spring 2024 Answer Key

1. $$\frac{d}{dx}\ln\left(\frac{\sqrt{1+e^x}\cdot\ln(3x)}{e^{\tan x}(4-x^5)^6}\right) = \frac{d}{dx}\ln\left(\sqrt{1+e^x}\cdot\ln(3x)\right) - \ln\left(e^{\tan x}(4-x^5)^6\right)$$

$$= \frac{d}{dx}\ln\left((1+e^x)^{1/2}\right) + \ln(\ln(3x)) - \left[\cancel{\ln}(e^{\tan x}) + \ln\left((4-x^5)^6\right)\right]$$

$$= \frac{d}{dx}\frac{1}{2}\ln(1+e^x) + \ln(\ln(3x)) - \tan x - 6\ln(4-x^5)$$

$$\boxed{= \frac{1}{2}\cdot\frac{1}{1+e^x}\cdot e^x + \frac{1}{\ln(3x)}\cdot\frac{1}{\cancel{3}x}\cdot\cancel{3} - \sec^2 x - 6\cdot\frac{1}{4-x^5}\cdot(-5x^4)}$$

2. $y=(\cos x)^x$ Logarithmic Differentiation

$$\ln y = \ln\left((\cos x)^x\right) = x\cdot\ln(\cos x)$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left(x\cdot\ln(\cos x)\right)$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = x\cdot\frac{1}{\cos x}\cdot(-\sin x) + \ln(\cos x)\cdot 1$$

$$\frac{dy}{dx} = \overset{(\cos x)^x}{\cancel{y}}\left(\overset{\tan x}{\frac{-x\,\cancel{\sin x}}{\cancel{\cos x}}} + \ln(\cos x)\right)$$

$$= \boxed{(\cos x)^x\left(-x\tan x + \ln(\cos x)\right)}$$

3(a) $$f(x) = \ln(1+\cos x) - e^{\sin x} - \overset{\text{constant}}{\ln 5}$$

$$f'(x) = \frac{1}{1+\cos x}\cdot(-\sin x) - e^{\sin x}\cdot\cos x + 0$$

$$f'(0) = \frac{\overset{1}{\cancel{1}}}{\underset{1}{\cancel{1+\cos 0}}}\cdot(-\overset{0}{\cancel{\sin 0}}) - \overset{1}{\cancel{e^{\overset{0}{\cancel{\sin 0}}}}}\cdot\overset{1}{\cancel{\cos 0}}$$

$$= 1\cdot 0 - 1\,1 = 0-1 = -1$$

3(b) $$f(x) = \cos(\ln(1+x)) - e^x + e^{(e^x)} + \frac{e}{1+\ln(x+1)}$$

$e(1+\ln(x+1))^{-1}$

$$f'(x) = -\sin(\ln(x+1))\cdot\frac{1}{x+1} - e^x + e^{(e^x)}\cdot e^x - e(1+\ln(x+1))^{-2}\cdot\frac{1}{x+1}$$

$$f'(0) = -\overset{0}{\cancel{\sin}}\left(\overset{0}{\cancel{\ln(0+1)}}\right)\cdot\overset{1}{\cancel{\frac{1}{0+1}}} - \overset{-1}{\cancel{e^0}} + \overset{e}{\cancel{e^{(e^0)}}}\cdot\overset{1}{\cancel{e^0}} - \overset{-e}{\cancel{\frac{e}{(1+\ln(1+0))^2}}}\cdot\overset{1}{\cancel{\frac{1}{0+1}}}$$

$$= 0\cdot 1 - 1 + \cancel{e} - \cancel{e} = -1$$

4(a) $f(x) = e^{8x} + \underset{8e^{-8x}}{\dfrac{8}{e^{8x}}} + \underset{\text{constant } 8}{\cancel{e}^{\ln 8}} - \underset{-8x^{-1}}{\dfrac{8}{x}} + \underset{e^{-7x}}{\dfrac{e^x}{e^{8x}}}$

$f'(x) = 8e^{8x} - 64e^{-8x} + 0 + 8x^{-2} - 7e^{-7x}$

4(b) $f(x) = 8e^x + 8e^{-x} - \underset{\text{constant } 8}{\cancel{\ln(e^8)}} - \underset{-ex^{-8}}{\dfrac{e}{x^8}} + \underset{e^{9x}}{e^{8x} \cdot e^x}$

k-rule $\int e^{kx}\,dx = \dfrac{e^{kx}}{k} + C$ (constant)

$\int f(x)\,dx = 8e^x + \dfrac{8e^{-x}}{-1} - 8x \cancel{-} \dfrac{ex^{-7}}{\cancel{-}7} + \dfrac{e^{9x}}{9} + C$

$= 8e^x - \dfrac{8}{e^x} - 8x + \dfrac{e}{7x^7} + \dfrac{e^{9x}}{9} + C$

5(a) $\int \dfrac{(1+e^{3x})^2}{e^{3x}}\,dx = \int \dfrac{1+2e^{3x}+e^{6x}}{e^{3x}}\,dx = \int \dfrac{1}{e^{3x}} + \dfrac{2\cancel{e^{3x}}}{\cancel{e^{3x}}} + \dfrac{e^{6x}}{e^{3x}}\,dx$ (FOIL)

$= \int e^{-3x} + 2 + e^{3x}\,dx = \dfrac{e^{-3x}}{-3} + 2x + \dfrac{e^{3x}}{3} + C$

5(b) $\int \dfrac{e^{3x}}{(1+e^{3x})^2}\,dx = \dfrac{1}{3}\int \dfrac{1}{u^2}\,du = \dfrac{1}{3}\int u^{-2}\,du = \dfrac{1}{3}\cdot\dfrac{u^{-1}}{-1} + C = -\dfrac{1}{3(1+e^{3x})} + C$

$$\boxed{\begin{aligned} u &= 1+e^{3x} \\ du &= 3e^{3x}\,dx \\ \tfrac{1}{3}\,du &= e^{3x}\,dx \end{aligned}}$$

5(c) $\int \dfrac{1}{\sqrt{x}\,e^{\sqrt{x}}}\,dx = 2\int \dfrac{1}{e^u}\,du = 2\int e^{-u}\,du = 2\cdot\dfrac{e^{-u}}{-1} + C = -\dfrac{2}{e^{\sqrt{x}}} + C$

$$\boxed{\begin{aligned} u &= \sqrt{x} \\ du &= \frac{1}{2\sqrt{x}}\,dx \\ 2\,du &= \frac{1}{\sqrt{x}}\,dx \end{aligned}}$$

5(d) $\int \dfrac{5-x^8}{x^9}\,dx = \int \dfrac{5}{x^9} - \dfrac{x^8}{x^9}\,dx = \int 5x^{-9} - \dfrac{1}{x}\,dx = \dfrac{5x^{-8}}{-8} - \ln|x| + C = -\dfrac{5}{8x^8} - \ln|x| + C$

5(e) $\int \dfrac{x^8}{5-x^9}\,dx = -\dfrac{1}{9}\int \dfrac{1}{u}\,du = -\dfrac{1}{9}\ln|u| + C = -\dfrac{1}{9}\ln|5-x^9| + C$

$$\boxed{\begin{aligned} u &= 5-x^9 \\ du &= -9x^8\,dx \\ -\tfrac{1}{9}\,du &= x^8\,dx \end{aligned}}$$

6(a) $\int_0^{\frac{\pi}{3}} \tan x\,dx = \int_0^{\frac{\pi}{3}} \frac{\sin x}{\cos x}\,dx = -\int_1^{\frac{1}{2}} \frac{1}{u}\,du = -\ln|u|\Big|_1^{\frac{1}{2}}$

$= -\ln\left|\tfrac{1}{2}\right| - \left(-\cancel{\ln|1|}^{0}\right)$

$= -\ln\frac{1}{2} = -\left(\cancel{\ln 1}^{0} - \ln 2\right) = \ln 2$

$\boxed{\begin{array}{l} u = \cos x \\ du = -\sin x\,dx \\ -du = \sin x\,dx \end{array}}$ $\boxed{\begin{array}{l} x=0 \Rightarrow u = \cos 0 = 1 \\ x = \frac{\pi}{3} \Rightarrow u = \cos\frac{\pi}{3} = \frac{1}{2} \end{array}}$

6(b) $\int_0^{\ln 2} \frac{e^{3x}}{\sqrt{8+e^{3x}}}\,dx = \frac{1}{3}\int_9^{16} \frac{1}{\sqrt{u}}\,du = \frac{1}{3}\int_9^{16} u^{-\frac{1}{2}}\,du = \frac{1}{3}\cdot\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\Bigg|_9^{16}$

$= \frac{2}{3}\sqrt{u}\Big|_9^{16} = \frac{2}{3}\left(\cancel{\sqrt{16}}^{4} - \cancel{\sqrt{9}}^{3}\right) = \frac{2}{3}\cdot 1 = \frac{2}{3}$

$\boxed{\begin{array}{l} u = 8+e^{3x} \\ du = 3e^{3x}dx \\ \frac{1}{3}du = e^{3x}dx \end{array}}$ $\boxed{\begin{array}{l} x=0 \Rightarrow u = 8+\cancel{e^0}^{1} = 9 \\ x=\ln 2 \Rightarrow u = 8+e^{3\ln 2} \\ \quad = 8+e^{\ln(2^3)} \\ \quad = 8+8 = 16 \end{array}}$

6(c) $\int_e^{e^5} \frac{1}{x(3+\ln x)}\,dx = \int_4^8 \frac{1}{u}\,du = \ln|u|\Big|_4^8 = \ln 8 - \ln 4 = \ln\left(\frac{8}{4}\right) = \ln 2$

$\boxed{\begin{array}{l} u = 3+\ln x \\ du = \frac{1}{x}dx \end{array}}$ $\boxed{\begin{array}{l} x=e \Rightarrow u = 3+\cancel{\ln e}^{1} = 4 \\ x=e^5 \Rightarrow u = 3+\cancel{\ln(e^5)}^{5} = 8 \end{array}}$

Bonus

$\lim_{n\to\infty} \frac{e^{1+\frac{1}{n}} + e^{1+\frac{2}{n}} + e^{1+\frac{3}{n}} + \cdots + e^2}{n} = \lim_{n\to\infty} \underbrace{\frac{1}{n}}_{\Delta x} \sum_{i=1}^{n} e^{1+\frac{i}{n}} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\cdot\Delta x$

Option 1: Riemann Sum for $f(x) = e^x$, $a=1 \Rightarrow b=2$ since looks like $\Delta x = \frac{1}{n} = \frac{b-a}{n}$ (2, 1)

$= \int_1^2 e^x\,dx = e^x\Big|_1^2 = e^2 - e$

Option 2: OR, Riemann Sum for $f(x) = e^{1+x}$, $a=0 \Rightarrow b=1$ since looks like $\Delta x = \frac{1}{n} = \frac{b-a}{n}$ (1, 0)

$= \int_0^1 e^{x+1}\,dx = \int_1^2 e^u\,du = e^u\Big|_1^2 = e^2 - e^1$

$\boxed{\begin{array}{l} u = x+1 \\ du = dx \end{array}}$ $\boxed{\begin{array}{l} x=0 \Rightarrow u=1 \\ x=1 \Rightarrow u=2 \end{array}}$