Name: Answer Key

Amherst College

DEPARTMENT OF MATHEMATICS

Math 106

Midterm Exam #3

April 19, 2017

• This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, webpages, or other aids are permitted.

• Simplify numerical answers such as $\sin\left(\frac{\pi}{6}\right)$, $\ln(e^3)$, $e^{2\ln 3}$ and $4^{\frac{3}{2}}$.

• Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

Problem	Score	Possible Points
1		30
2		10
3		54
4		6
Total		100

1. [30 Points] Compute each of the following derivatives.

(a) y' where $y = \ln\left(\dfrac{(\sin^2 x)\,\sqrt{1+\sec\sqrt{x}}}{(5-x^7)^{\frac{3}{7}}\,e^{-\cos x}}\right)$ Do not simplify your final answer here.

Simplify $= \ln\left[\sin^2 x \cdot \sqrt{1+\sec\sqrt{x}}\right] - \ln\left[(5-x^7)^{3/2}\, e^{-\cos x}\right]$

$-\cos x$

$= \ln(\sin^2 x) + \ln\sqrt{1+\sec\sqrt{x}} - \left[\ln\left[(5-x^7)^{3/2}\right] + \cancel{\ln e}^{-\cos x}\right]$

$= 2\ln(\sin x) + \frac{1}{2}\ln(1+\sec\sqrt{x}) - \frac{3}{2}\ln(5-x^7) + \cos x$

$$\Rightarrow \boxed{y' = \frac{2}{\sin x}\cdot\cos x + \frac{1}{2}\cdot\frac{1}{(1+\sec\sqrt{x})}\cdot\sec\sqrt{x}\tan\sqrt{x}\cdot\frac{1}{2\sqrt{x}} - \frac{3}{7}\left(\frac{1}{5-x^7}\right)(-7x^6) - \sin x}$$

(b) $\dfrac{d}{dx}(\tan x)^{\sqrt{x}}$

Set $y = (\tan x)^{\sqrt{x}}$

$\ln y = \ln\left[(\tan x)^{\sqrt{x}}\right] = \sqrt{x}\,\ln(\tan x)$

$\frac{d}{dx}\left[\ln y\right] = \frac{d}{dx}\left[\sqrt{x}\,\ln(\tan x)\right]$

$\frac{1}{y}\frac{dy}{dx} = \sqrt{x}\cdot\frac{1}{\tan x}\cdot\sec^2 x + \ln(\tan x)\cdot\frac{1}{2\sqrt{x}}$

$(\tan x)^{\sqrt{x}}$

Solve $\frac{dy}{dx} = y\left[\frac{\sqrt{x}\sec^2 x}{\tan x} + \frac{\ln(\tan x)}{2\sqrt{x}}\right]$

1

$$\boxed{= (\tan x)^{\sqrt{x}}\left[\frac{\sqrt{x}\sec^2 x}{\tan x} + \frac{\ln(\tan x)}{2\sqrt{x}}\right]}$$

1. (Continued) Compute the following derivative.

(c) $\dfrac{dy}{dx}$ where $y = \dfrac{1}{\sqrt{e^{7+\cos x}}} + \dfrac{1}{e^{\sqrt{5x+2}}} + \sqrt{\ln(\tan x)} + \dfrac{1}{\ln\sqrt{1+\sec^2 x}}$

Can also simplify power

$$\text{Prep } y = \left[e^{7+\cos x}\right]^{-1/2} + e^{-\sqrt{5x+2}} + \left[\ln(\tan x)\right]^{1/2} + \left[\ln\sqrt{1+\sec^2 x}\right]^{-1}$$

$$\frac{dy}{dx} = -\frac{1}{2}\left[e^{7+\cos x}\right]^{-3/2} e^{7+\cos x}\cdot(-\sin x) + e^{-\sqrt{5x+2}}\cdot\left(-\frac{1}{2\sqrt{5x+2}}\right)\cdot 5 \cdots$$

continued.

$$\cdots + \frac{1}{2\sqrt{\ln(\tan x)}}\cdot\frac{1}{\tan x}\cdot\sec^2 x - \left(\ln\sqrt{1+\sec^2 x}\right)^{-2}\cdot\frac{1}{\sqrt{1+\sec^2 x}}\cdot\frac{1}{2\sqrt{1+\sec^2 x}}\cdot 2\sec x\cdot\sec x\tan x$$

2. [10 Points] Find the equation of the tangent line to the curve

$$y = \ln(1+\cos x) + \cos(\ln(1+x)) - e^{\sin x} + \frac{e}{1+\ln(x+1)} + \underbrace{[e^{x+1}\cdot\cos(e^x - 1)]}_{\text{product.}} - \ln 2$$

at the point where $x = 0$.

y-value:

$$y(0) = \ln(1+\cancel{\cos 0}^{\,1}) + \cancel{\cos(\ln(1+0))}^{\,1} - \cancel{e^{\sin 0}}^{\,1} + \frac{e}{1+\ln(0+1)} + e\cdot\cancel{\cos(e^0-1)}^{\,1} - \ln 2.$$

$$= \ln 2 + 1 - 1 + e + e - \cancel{\ln 2}$$

$$= 2e.$$

Derivative:

$$y' = \frac{1}{1+\cos x}\cdot(-\sin x) - \sin(\ln(1+x))\cdot\frac{1}{1+x}(1) - e^{\sin x}\cdot\cos x - \frac{e}{[1+\ln(x+1)]^2}\cdot\frac{1}{x+1}\ \cdots$$

continued.

$$+\cdots+ \underbrace{e^{x+1}\cdot(-\sin(e^x-1))\cdot e^x + \cos(e^x-1)\,e^{x+1}(1)}_{\text{product rule.}} - 0$$

Specific Slope:

$$y'(0) = \cancel{\frac{1}{1+\cos 0}(-\sin 0)}^{\,0} - \frac{\cancel{\sin(\ln 1)}^{\,0}}{1} - \cancel{e^{\sin 0}}^{\,1}\cdot\cancel{\cos 0}^{\,1} - \cancel{\frac{e}{(1+\ln 1)^2}}^{\,-e}\cdot(1) + \cancel{e^1(-\sin(e^0-1))e^0}^{\,0} + \cancel{\cos(e^0-1)}^{\,1}e^1$$

$$= 0 + 0 - 1 - \cancel{e} + 0 + \cancel{e} = -1$$

Point Slope Form: $y - 2e = -1(x-0)$

Finally, $\boxed{y = -x + 2e}$

3. [54 Points] Evaluate each of the following integrals. Simplify.

(a) $\displaystyle\int_0^{\ln 2}\left(e^x+\frac{1}{e^x}\right)\left(1+\frac{1}{e^{2x}}\right)\,dx = \int_0^{\ln 2} e^x + \frac{e^x}{e^{2x}} + \frac{1}{e^x} + \frac{1}{e^{3x}}\,dx$

$$= \int_0^{\ln 2} e^x + \frac{1}{e^x} + \frac{1}{e^x} + \frac{1}{e^{3x}}\,dx = \int_0^{\ln 2} e^x + 2e^{-x} + e^{-3x}\,dx$$

$$= e^x + \frac{2e^{-x}}{(-1)} + \frac{e^{-3x}}{(-3)}\Bigg|_0^{\ln 2} = e^x - \frac{2}{e^x} - \frac{1}{3e^{3x}}\Bigg|_0^{\ln 2}$$

$$= \cancel{e^{\ln 2}}^{\,2} - \frac{2}{\underbrace{e^{\ln 2}}_{2}} - \frac{1}{3\underbrace{e^{3\ln 2}}_{8}} - \left(\underbrace{e^0}_{1} - \frac{2}{\underbrace{e^0}_{1}} - \frac{1}{3\underbrace{e^0}_{1}}\right) = 2 - \cancel{1} - \frac{1}{24} \overbrace{- \cancel{1} + 2}^{\cancel{1}} + \frac{1}{3}$$

$$= 2 - \frac{1}{24} + \frac{1}{3} = \frac{48}{24} - \frac{1}{24} + \frac{8}{24} = \boxed{\frac{55}{24}}$$

(b) $\displaystyle\int_1^{\sqrt{3}} \frac{w}{4-w^2}\,dw = -\frac{1}{2}\int_3^1 \frac{1}{u}\,du = -\frac{1}{2}\ln|u|\,\Bigg|_3^1$

$$u = 4 - w^2$$
$$du = -2w\,dw$$
$$-\tfrac{1}{2}\,du = w\,dw$$

$w = 1 \Rightarrow u = 4 - 1 = 3$

$w = \sqrt{3} \Rightarrow u = 4 - (\sqrt{3})^2 = 1$

$$= -\tfrac{1}{2}\left[\cancel{\ln 1}^{\,0} - \ln 3\right]$$

$$= \boxed{\frac{\ln 3}{2}}$$

or $= -\dfrac{1}{2}\ln\left(\dfrac{1}{3}\right)$.

or $= \ln\left(3^{1/2}\right) = \ln\sqrt{3}$.

3. (Continued) Evaluate each of the following integrals. Simplify.

(c) $\int_1^{e^3} \frac{\sqrt{4-\ln x}}{x}\,dx = -\int_4^1 \sqrt{u}\,du = -\frac{2}{3} u^{3/2} \Big|_4^1$

$u^{1/2}$

$$\boxed{\begin{aligned} u &= 4-\ln x \\ du &= -\tfrac{1}{x}\,dx \\ -du &= \tfrac{1}{x}\,dx \end{aligned}}$$

$$= -\frac{2}{3}\left[\cancel{1^{3/2}}^{\,1} - \cancel{4^{3/2}}^{\,8}\right]$$

$4^{3/2} = (\sqrt{4})^3 = 8$

$$= -\frac{2}{3}\left[1-8\right] = \boxed{\frac{14}{3}}$$

$$\boxed{\begin{aligned} x=1 &\Rightarrow u = 4-\cancel{\ln 1}^{\,0} = 4 \\ x=e^3 &\Rightarrow u = 4-\cancel{\ln(e^3)}_{\,3} = 1 \end{aligned}}$$

SPLIT

(d) $\int_{-e}^{-1} \frac{1-x^2}{x}\,dx = \int_{-e}^{-1} \frac{1}{x} - \frac{x^2}{x}\,dx = \int_{-e}^{-1} \frac{1}{x} - x\,dx$

$$= \ln|x| - \frac{x^2}{2}\Big|_{-e}^{-1} = \left(\ln\cancel{|-1|}^{\,1} - \frac{(-1)^2}{2}\right) - \left(\ln\cancel{|-e|}^{\,e} - \frac{(-e)^2}{2}\right)$$

Need Absolute Values

Needed

Needed

$$= 0 - \tfrac{1}{2} - \left(1 - \tfrac{e^2}{2}\right)$$

$$= \frac{-1}{2} - 1 + \frac{e^2}{2}$$

$$= \frac{-3}{2} + \frac{e^2}{2}$$

$$= \boxed{\frac{e^2}{2} - \frac{3}{2}}$$

3. (Continued) Evaluate each of the following integrals. Simplify.

(e) $\int \frac{\left(x^{\frac{3}{4}}-1\right)\left(x^3-x^{\frac{5}{4}}\right)}{x^3}\,dx$ $\int \frac{x^{3/4}\cdot x^3 - x^{3/4}x^{5/4} - x^3 + x^{5/4}}{x^3}\,dx$

$= \int \frac{x^{15/4} - \cancel{x^{8/4}} - x^3 + x^{5/4}}{x^3}\,dx = \int \frac{x^{15/4}}{x^3} - \frac{x^2}{x^3} - \frac{x^3}{x^3} + \frac{x^{5/4}}{x^3}\,dx$

x^2

SPLIT

$= \int x^{3/4} - \frac{1}{x} - 1 + x^{-7/4}\,dx = \boxed{\frac{4}{7}x^{7/4} - \ln|x| - x - \frac{4}{3}x^{-3/4} + C}$

(f) $\int_1^4 \frac{1}{\sqrt{x}\,e^{1+\sqrt{x}}}\,dx = 2\int_2^3 \frac{1}{e^u}\,du = 2\int_2^3 e^{-u}\,du = \left.\frac{2e^{-u}}{(-1)}\right|_2^3$

$\boxed{\begin{array}{l} u = 1+\sqrt{x} \\ du = \frac{1}{2\sqrt{x}}\,dx \\ 2du = \frac{1}{\sqrt{x}}\,dx \end{array}}$

$= \left.-\frac{2}{e^u}\right|_2^3 = \frac{-2}{e^3} + \frac{2}{e^2}$

$\boxed{\begin{array}{l} x=1 \Rightarrow u = 1+\sqrt{1} = 2 \\ x=4 \Rightarrow u = 1+\sqrt{4} = 3 \end{array}}$

$= \boxed{\frac{2}{e^2} - \frac{2}{e^3}}$

3. (Continued) Evaluate the following integral. Simplify.

(g) $\int_0^{\frac{\pi}{9}} \tan(3x)\,dx$ $= \int_0^{\pi/9} \frac{\sin(3x)}{\cos(3x)}\,dx = -\frac{1}{3}\int_1^{1/2} \frac{1}{u}\,du = -\frac{1}{3} \ln|u| \Big|_1^{1/2}$

$$\boxed{\begin{array}{l} u=\cos(3x) \\ du=-3\sin(3x)dx \\ -\frac{1}{3}du=\sin(3x)dx \end{array}}$$

$$= -\frac{1}{3}\left[\ln\left(\frac{1}{2}\right) - \cancel{\ln 1}^{\,0}\right] = \boxed{-\frac{1}{3}\ln\left(\frac{1}{2}\right)}$$

$$\boxed{\begin{array}{l} x=0 \Rightarrow u=\cos 0=1 \\ x=\pi/9 \Rightarrow u=\cos(\pi/3)=1/2 \end{array}}$$

4. [6 Points] Find the function $f(x)$ that satsifies $f'(x) = \frac{e^x}{3e^x - e^5}\,dx$ and $f(5) = 2$.

$$f(x) = \int \frac{e^x}{3e^x - e^5}\,dx = \frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|3e^x - e^5| + C$$

$$\boxed{\begin{array}{l} u=3e^x-e^5 \\ du=3e^x\,dx \\ \frac{1}{3}du=e^x dx \end{array}}$$

Test

$$f(5) = \frac{1}{3}\ln|\underbrace{3e^5 - e^5}_{2e^5}| + C \overset{set}{=} 2.$$

Solve $C = 2 - \frac{1}{3}\ln(2e^5)$

Finally, $f(x) = \boxed{\frac{1}{3}\ln|3e^x - e^5| + \left(2 - \frac{1}{3}\ln(2e^5)\right)}$