Exam 2 Spring 2025 Answer Key

1 FTC $\int_{-1}^{2} 2-3x-x^2\,dx = 2x - \frac{3x^2}{2} - \frac{x^3}{3}\Big|_{-1}^{2} = 4 - \frac{12}{2} - \frac{8}{3} - \left(-2 - \frac{3}{2} + \frac{1}{3}\right)$

$= \cancel{4} - \cancel{6} - \frac{8}{3} + \cancel{2} + \frac{3}{2} - \frac{1}{3} = \frac{3}{2} - \frac{\cancel{9}^{3}}{\cancel{3}} = \frac{3}{2} - 3 = \frac{3}{2} - \frac{6}{2} = \frac{-3}{2}$

Cancel!

Limit Definition

$$\int_{-1}^{2} 2-3x-x^2\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\cdot\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(-1+\frac{3i}{n}\right)\cdot\left(\frac{3}{n}\right)$$

$a=-1$, $b=2$ $\Big\rangle$ $\Delta x = \frac{b-a}{n} = \frac{2-(-1)}{n} = \frac{3}{n}$

$x_i = a + i\Delta x = -1 + i\left(\frac{3}{n}\right) = -1 + \frac{3i}{n}$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} 2 - 3\left(-1+\frac{3i}{n}\right) - \left(-1+\frac{3i}{n}\right)^2$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} 2 + 3 - \frac{9i}{n} - \left(1 - \frac{6i}{n} + \frac{9i^2}{n^2}\right)$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} 2 + 3 - \frac{9i}{n} - 1 + \frac{6i}{n} - \frac{9i^2}{n^2}$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} 4 - \frac{3i}{n} - \frac{9i^2}{n^2}$$

pull all non-i values out of Series Sums

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} 4 - \frac{3}{n} \sum_{i=1}^{n} \frac{3i}{n} - \frac{3}{n} \sum_{i=1}^{n} \frac{9i^2}{n^2}$$

$$= \lim_{n\to\infty} \frac{12}{n} \cancel{\sum_{i=1}^{n} 1}^{\,n} - \frac{9}{n^2} \sum_{i=1}^{n} i - \frac{27}{n^3} \sum_{i=1}^{n} i^2$$

i-Formulas

repartner

$$= \lim_{n\to\infty} \frac{12}{\cancel{n}} \cdot \cancel{n} - \frac{9}{n^2}\left(\frac{n(n+1)}{2}\right) - \frac{27}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$$

$$= \lim_{n\to\infty} 12 - \frac{9}{2}\left(\frac{\cancel{n}}{\cancel{n}}\right)\left(\frac{n+1}{n}\right) - \frac{27}{6}\left(\frac{\cancel{n}}{\cancel{n}}\right)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)$$

$$= \lim_{n\to\infty} 12 - \frac{9}{2}(1)\left(\frac{\cancel{n}}{\cancel{n}} + \frac{1}{n}\right) - \frac{27}{6}(1)\left(\frac{\cancel{n}}{\cancel{n}} + \frac{1}{n}\right)\left(\frac{2\cancel{n}}{\cancel{n}} + \frac{1}{n}\right)$$

$$= \lim_{n\to\infty} 12 - \frac{9}{2}\left(1 + \cancel{\frac{1}{n}}^{\,0}\right) - \frac{27}{6}\left(1 + \cancel{\frac{1}{n}}^{\,0}\right)\left(2 + \cancel{\frac{1}{n}}^{\,0}\right)$$

Finally ⟿ Let $n\to\infty$

$$= 12 - \frac{9}{2} - \frac{27}{\cancel{6}_{3}}(1)(\cancel{2}) = 12 - \frac{9}{2} - \frac{\cancel{27}^{9}}{\cancel{3}}$$

$$= 12 - \frac{9}{2} - 9 = 3 - \frac{9}{2} = \frac{6}{2} - \frac{9}{2} = \frac{-3}{2}$$ Match!

2(a) $\displaystyle\int \frac{x^6}{\sqrt{3x^7+5}}\,dx = \frac{1}{21}\int \frac{1}{\sqrt{u}}\,du = \frac{1}{21}\int u^{-1/2}\,du = \frac{1}{21}\frac{u^{1/2}}{1/2} + C$

$$\boxed{\begin{aligned} u &= 3x^7+5 \\ du &= 21x^6\,dx \\ \tfrac{1}{21}\,du &= x^6\,dx \end{aligned}}$$

$= \frac{2}{21}\sqrt{u} + C = \frac{2}{21}\sqrt{3x^7+5} + C$

2(b) $\displaystyle\int \frac{x}{(x+5)^4}\,dx = \int \frac{u-5}{u^4}\,du = \int \frac{u}{u^4} - \frac{5}{u^4}\,du = \int u^{-3} - 5u^{-4}\,du$

reverse/invert

$$\boxed{\begin{aligned} u &= x+5 \Rightarrow x = u-5 \\ du &= dx \end{aligned}}$$

$= \frac{u^{-2}}{-2} \cancel{-}\, 5\left(\frac{u^{-3}}{\cancel{-}3}\right) + C = -\frac{1}{2u^2} + \frac{5}{3u^3} + C$

$= -\frac{1}{2(x+5)^2} + \frac{5}{3(x+5)^3} + C$

3(a) $\displaystyle\int_{\pi/6}^{\pi/3} \frac{\sin x}{\cos^5 x}\,dx = \int_{\pi/6}^{\pi/3} \frac{\sin x}{(\cos x)^5}\,dx = -\int_{\sqrt{3}/2}^{1/2} \frac{1}{u^5}\,du = \cancel{-}\left(\frac{u^{-4}}{\cancel{-}4}\right)\Bigg|_{\sqrt{3}/2}^{1/2} = \frac{1}{4u^4}\Bigg|_{\sqrt{3}/2}^{1/2}$

u^{-5}

$$\boxed{\begin{aligned} u &= \cos x \\ du &= -\sin x\,dx \\ -du &= \sin x\,dx \end{aligned}} \qquad \boxed{\begin{aligned} x = \tfrac{\pi}{6} &\Rightarrow u = \cos\tfrac{\pi}{6} = \tfrac{\sqrt{3}}{2} \\ x = \tfrac{\pi}{3} &\Rightarrow u = \cos\tfrac{\pi}{3} = \tfrac{1}{2} \end{aligned}}$$

$= \frac{1}{\cancel{4}\left(\frac{1}{2}\right)^4} - \frac{1}{\cancel{4}\left(\frac{\sqrt{3}}{2}\right)^4}$

$\cancel{1/16}$ 4 $\quad$ $\cancel{9/16}$ 4

$= 4 - \frac{4}{9} = \frac{36}{9} - \frac{4}{9} = \frac{32}{9}$

3(b) $\displaystyle\int_2^6 \frac{1}{x^2}\cdot\cos\left(\frac{\pi}{x}\right)dx = -\frac{1}{\pi}\int_{\pi/2}^{\pi/6} \cos u\,du = -\frac{1}{\pi}\sin u\Bigg|_{\pi/2}^{\pi/6}$

$$\boxed{\begin{aligned} u &= \tfrac{\pi}{x} = \pi x^{-1} \\ du &= -\pi x^{-2}\,dx \\ -\tfrac{1}{\pi}\,du &= \tfrac{1}{x^2}\,dx \end{aligned}} \qquad \boxed{\begin{aligned} x = 2 &\Rightarrow u = \tfrac{\pi}{2} \\ x = 6 &\Rightarrow u = \tfrac{\pi}{6} \end{aligned}}$$

$= -\frac{1}{\pi}\left(\underbrace{\cancel{\sin\tfrac{\pi}{6}}}_{\frac{1}{2}} - \underbrace{\cancel{\sin\tfrac{\pi}{2}}}_{1}\right)$

$= \cancel{-}\frac{1}{\pi}\left(\cancel{-}\frac{1}{2}\right) = \frac{1}{2\pi}$

3(c) $\displaystyle\int_1^4 \frac{\frac{1}{x}+1}{\sqrt{x}}\,dx = \int_1^4 \frac{\frac{1}{x}}{\sqrt{x}} + \frac{1}{\sqrt{x}}\,dx = \int_1^4 x^{-\frac{3}{2}} + x^{-\frac{1}{2}}\,dx = \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}}\Bigg|_1^4$

$\displaystyle = \frac{-2}{\sqrt{x}} + 2\sqrt{x}\,\Bigg|_1^4 = \frac{-2}{\cancel{\sqrt{4}}_{\,2}} + 2\cancel{\sqrt{4}}^{\,2} - \left(\frac{-2}{\cancel{\sqrt{1}}_{\,1}} + 2\cancel{\sqrt{1}}^{\,1}\right) = -1 + 4 + \cancel{2} - \cancel{2} = 3$

3(d) $\displaystyle\int_4^9 \frac{3}{\sqrt{x}\,(7+\sqrt{x})^2}\,dx = 2\int_9^{10} \frac{3}{u^2}\,du = 6\int_9^{10} u^{-2}\,du = 6\left(\frac{u^{-1}}{-1}\right)\Bigg|_9^{10}$

$\boxed{\begin{array}{l} u = 7+\sqrt{x} \\ du = \frac{1}{2\sqrt{x}}\,dx \\ 2du = \frac{1}{\sqrt{x}}\,dx \end{array}}$

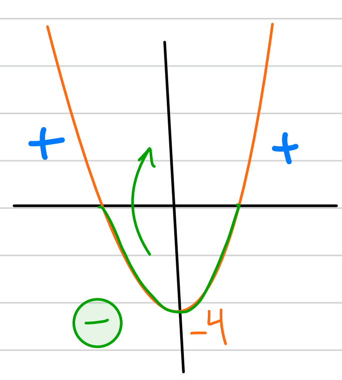
$\boxed{\begin{array}{l} x = 4 \Rightarrow u = 7+\cancel{\sqrt{4}}^{\,2} = 9 \\ x = 9 \Rightarrow u = 7+\cancel{\sqrt{9}}^{\,3} = 10 \end{array}}$

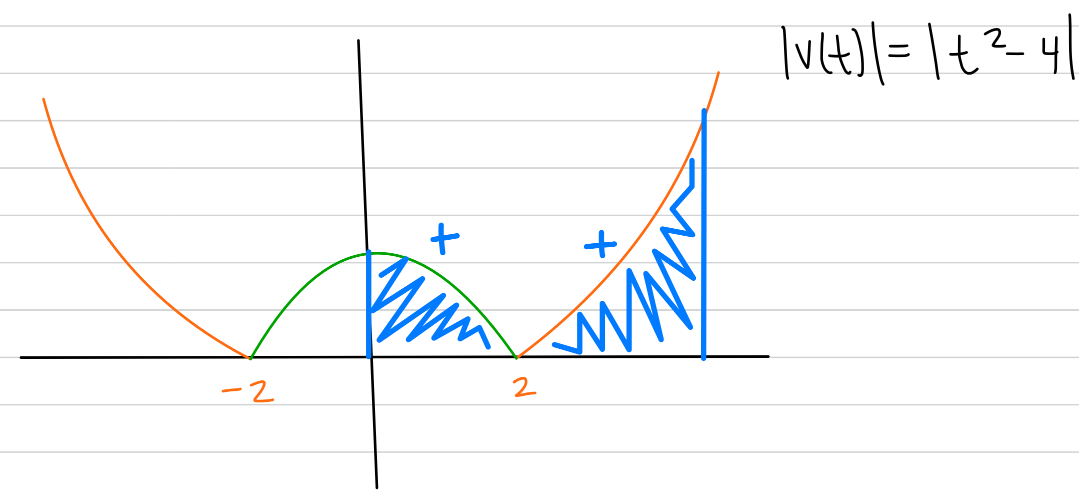
$\displaystyle = -\frac{6}{u}\Bigg|_9^{10} = -\frac{6}{10} + \frac{6}{9}$

$\displaystyle = \frac{-54}{90} + \frac{60}{90} = \frac{6}{90} = \frac{3}{45} = \frac{1}{15}$

4. $v(t) = t^2 - 4$ $\qquad t^2 - 4 = 0 \Rightarrow t^2 = 4 \quad t = \pm 2$

Graph $|v(t)| = |t^2-4|$





Cases $|t^2-4| = \begin{cases} t^2-4 & \text{if } t \le -2 \text{ or } t \ge 2 \\ -(t^2-4) & \text{if } -2 < t < 2 \end{cases}$

Total Distance for $0 \le t \le 5$

$\displaystyle\int_0^5 |v(t)|\,dt = \int_0^2 |\overset{4-t^2}{\cancel{v(t)}}|\,dt + \int_2^5 |\overset{t^2-4}{\cancel{v(t)}}|\,dt$

$\displaystyle = 4t - \frac{t^3}{3}\Bigg|_0^2 + \frac{t^3}{3} - 4t\Bigg|_2^5$

$\displaystyle = 8 - \frac{8}{3} - (0-0) + \frac{125}{3} - 20 - \left(\frac{8}{3} - 8\right)$

$\displaystyle = 8 - \frac{8}{3} + \frac{125}{3} - 20 - \frac{8}{3} + 8$

$\displaystyle = -4 + \frac{109}{3} = \frac{109}{3} - \frac{12}{3} = \frac{97}{3}$ +

$\begin{array}{r} 125 \\ -16 \\ \hline 109 \end{array}$

5. $f(x) = \int f'(x)\,dx = \int \tan^3 x \cdot \sec^2 x\,dx = \int (\tan x)^3 \cdot \sec^2 x\,dx = \int u^3\,du$

$$= \frac{u^4}{4} + C = \frac{\tan^4 x}{4} + C$$

$$\boxed{\begin{array}{l} u = \tan x \\ du = \sec^2 x\,dx \end{array}}$$

Test value

$$(\sqrt{3})^4$$

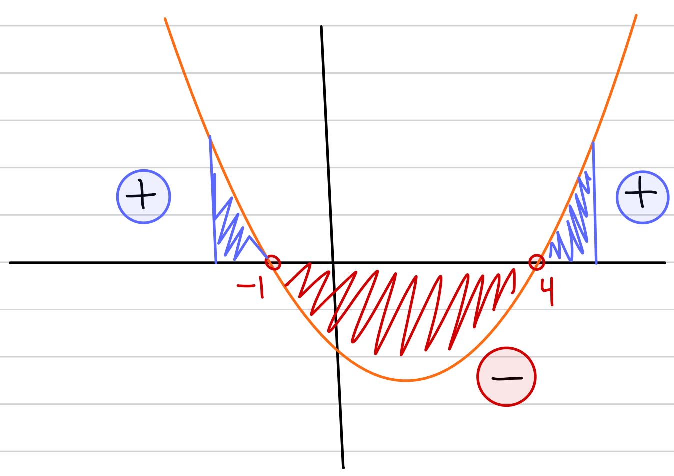
$$f\left(\frac{\pi}{3}\right) = \frac{1}{4}\left(\tan\left(\frac{\pi}{3}\right)\right)^4 + C \overset{\text{Set}}{=} \frac{1}{4}$$

$$\frac{9}{4} + C = \frac{1}{4} \Rightarrow C = \frac{1}{4} - \frac{9}{4} = \frac{-8}{4} = -2$$

Finally, $f(x) = \frac{\tan^4 x}{4} - 2$

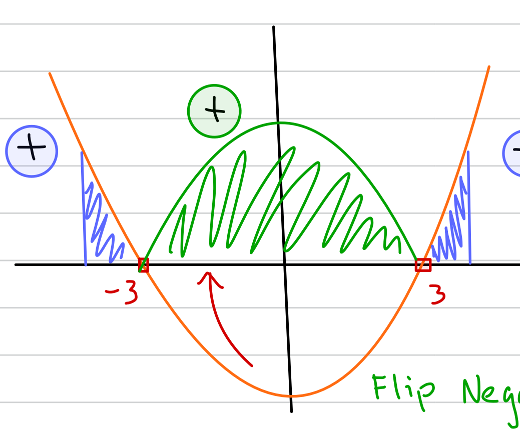
6(a) $\int_{-2}^{5} x^2 - 3x - 4\,dx$ Factor $x^2 - 3x - 4 = (x-4)(x+1) = 0$

$x = 4$ or $x = -1$



6(b) $\int_{-4}^{4} |x^2 - 9|\,dx$ Factor $x^2 - 9 = 0$

$x^2 = 9 \Rightarrow x = \pm 3$



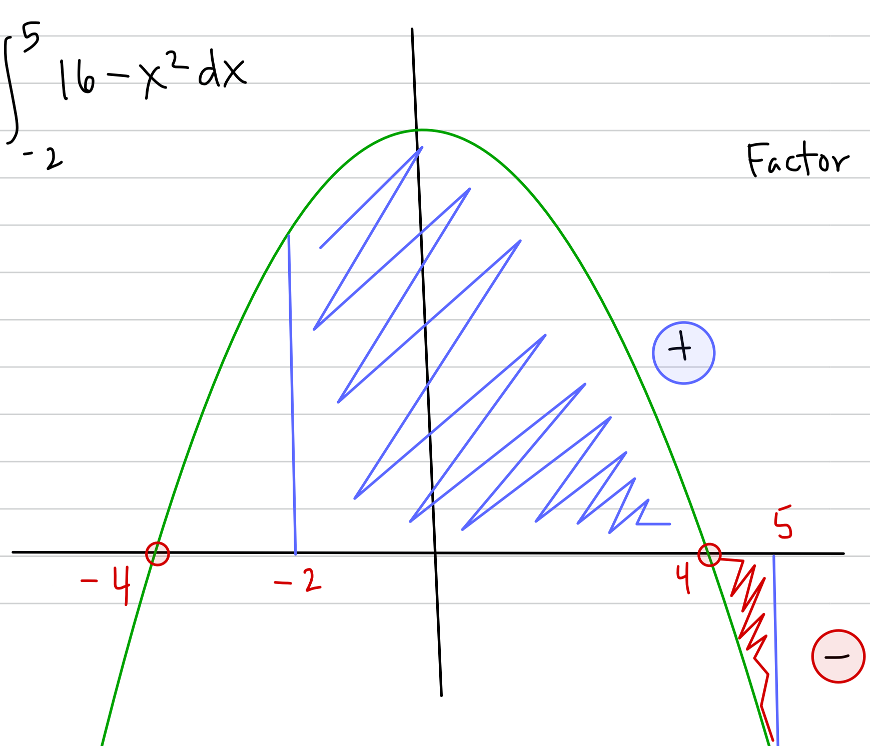
Flip Negative portion

6(c) $\int_{-2}^{5} 16 - x^2\,dx$

Factor $16 - x^2 = 0$

$x^2 = 16$

$x = \pm 4$



Bonus #1

$$(x^2)^4$$

$$\int \frac{x^9}{\sqrt{2-x^2}}\,dx = \int \frac{x^8 \cdot x}{\sqrt{2-x^2}}\,dx = -\frac{1}{2}\int \frac{(2-u)^4}{\sqrt{u}}\,du \qquad \text{Big FOIL}$$

$$\boxed{\begin{array}{l} u = 2-x^2 \Rightarrow x^2 = 2-u \\ du = -2x\,dx \\ -\frac{1}{2}\,du = x\,dx \end{array}}$$

$$= -\frac{1}{2}\int \frac{u^4 - 8u^3 + 24u^2 - 32u + 16}{\sqrt{u}}\,du \qquad \text{Split}$$

$$= -\frac{1}{2}\int u^{7/2} - 8u^{5/2} + 24u^{3/2} - 32u^{1/2} + 16u^{-1/2}\,du$$

$$= -\frac{1}{2}\left(\frac{u^{9/2}}{9/2} - 8\frac{u^{7/2}}{7/2} + 24\frac{u^{5/2}}{5/2} - 32\frac{u^{3/2}}{3/2} + 16\frac{u^{1/2}}{1/2}\right) + C$$

$$\frac{2}{9} \quad \frac{2}{7} \quad \frac{2}{5} \quad \frac{2}{3} \quad 2$$

$$= -\frac{1}{2}\left(\frac{2}{9}u^{9/2} - \frac{16}{7}u^{7/2} + \frac{48}{5}u^{5/2} - \frac{64}{3}u^{3/2} + 32\sqrt{u}\right) + C$$

$$(2-u)^2 = 4 - 4u + u^2$$

$$(2-u)^4 = (4-4u+u^2)(4-4u+u^2)$$

$$\begin{array}{l} = 16 - 16u + 4u^2 \\ \qquad - 16u + 16u^2 - 4u^3 \\ \qquad\qquad + 4u^2 - 4u^3 + u^4 \\ \hline 16 - 32u + 24u^2 - 8u^3 + u^4 \end{array}$$

$$= -\frac{1}{9}(2-x^2)^{9/2} + \frac{8}{7}(2-x^2)^{7/2} + \frac{24}{5}(2-x^2)^{5/2} + \frac{32}{3}(2-x^2)^{3/2} - 16\sqrt{2-x^2} + C$$