**Warm Up Algebra**
**for the Limit Definition of**
**the Definite Integral using Riemann Sums**

**Function Evaluation** For these problems, $i$ and $n$ represent some constants. Evaluate the specific function value here in terms of $i$ and $n$. Simplify the algebra by combining similar terms.

1. Consider the function $f(x) = 3x$. Compute $f\left(\frac{2i}{n}\right) = 3\left(\frac{2i}{n}\right) = \boxed{\frac{6i}{n}}$

2. Consider the function $f(x) = 3x$. Compute $f\left(5+\frac{2i}{n}\right) = 3\left(5+\frac{2i}{n}\right) = \boxed{15+\frac{6i}{n}}$

3. Consider $f(x) = 7x - 3$. Compute $f\left(\frac{4i}{n}\right) = 7\left(\frac{4i}{n}\right) - 3 = \boxed{\frac{28i}{n} - 3}$

4. Consider $f(x) = 7x - 3$.

Compute $f\left(8+\frac{4i}{n}\right) = 7\left(8+\frac{4i}{n}\right) - 3 = 56 + \frac{28i}{n} - 3 = \boxed{\frac{28i}{n} + 53}$

5. Consider $f(x) = x^2 + 6$. Compute $f\left(\frac{3i}{n}\right) = \left(\frac{3i}{n}\right)^2 + 6 = \boxed{\frac{9i^2}{n^2} + 6}$

6. Consider $f(x) = x^2 + 6$.

Compute $f\left(7+\frac{3i}{n}\right) = \left(7+\frac{3i}{n}\right)^2 + 6 = 49 + \frac{21i}{n} + \frac{21i}{n} + \frac{9i^2}{n^2} + 6 = \boxed{\frac{9i^2}{n^2} + \frac{42i}{n} + 55}$

7. Consider $f(x) = x^2 - 4x - 3$. Compute $f\left(\frac{5i}{n}\right) = \left(\frac{5i}{n}\right)^2 - 4\left(\frac{5i}{n}\right) - 3 = \boxed{\frac{25i^2}{n^2} - \frac{20i}{n} - 3}$

8. Consider $f(x) = x^2 - 4x - 3$.

Compute

$$f\left(-1+\frac{6i}{n}\right) = \left(-1+\frac{6i}{n}\right)^2 - 4\left(-1+\frac{6i}{n}\right) - 3$$

$$= 1 - \frac{6i}{n} - \frac{6i}{n} + \frac{36i^2}{n^2} + 4 - \frac{24i}{n} - 3 = \boxed{\frac{36i^2}{n^2} - \frac{36i}{n} + 2}$$

**Limit Evaluations** Practice evaluating limits that will arise in future problems.

1. $\lim\limits_{n\to\infty} 5 = \boxed{5}$

2. $\lim\limits_{n\to\infty} \frac{1}{n} = \lim\limits_{n\to\infty} \cancelto{0}{\frac{1}{\cancelto{\infty}{n}}} = \boxed{0}$

3. $\lim\limits_{n\to\infty} 1 + \frac{1}{n} = \lim\limits_{n\to\infty} 1 + \cancelto{0}{\frac{1}{\cancelto{\infty}{n}}} = 1 + 0 = \boxed{1}$

4. $\lim\limits_{n\to\infty} \frac{n+1}{n} \overset{\text{split}}{=} \lim\limits_{n\to\infty} \frac{n}{n} + \frac{1}{n} = \lim\limits_{n\to\infty} 1 + \frac{1}{n} = \lim\limits_{n\to\infty} 1 + \cancelto{0}{\frac{1}{\cancelto{\infty}{n}}} = 1 + 0 = \boxed{1}$

5. $\lim\limits_{n\to\infty} \frac{2n+1}{n} \overset{\text{split}}{=} \lim\limits_{n\to\infty} \frac{2n}{n} + \frac{1}{n} = \lim\limits_{n\to\infty} 2 + \frac{1}{n} = \lim\limits_{n\to\infty} 2 + \cancelto{0}{\frac{1}{\cancelto{\infty}{n}}} = 2 + 0 = \boxed{2}$

6.
$$\begin{aligned}
\lim_{n\to\infty} \frac{n(n+1)}{n^2} &= \lim_{n\to\infty} \frac{n(n+1)}{n \cdot n} \overset{\text{partner}}{=} \lim_{n\to\infty} \left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right) \\
&= \lim_{n\to\infty} (1)\left(\frac{n+1}{n}\right) \overset{\text{split}}{=} \lim_{n\to\infty} (1)\left(\frac{n}{n} + \frac{1}{n}\right) = \lim_{n\to\infty} (1)\left(1 + \frac{1}{n}\right) \\
&= \lim_{n\to\infty} (1)\left(1 + \cancelto{0}{\frac{1}{\cancelto{\infty}{n}}}\right) = (1)(1) = \boxed{1}
\end{aligned}$$

7.
$$\begin{aligned}
\lim_{n\to\infty} \frac{n(n+1)(2n+1)}{n^3} &= \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{n \cdot n \cdot n} \overset{\text{partner}}{=} \lim_{n\to\infty} \left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) \\
&= \lim_{n\to\infty} (1)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) \overset{\text{split}}{=} \lim_{n\to\infty} (1)\left(\frac{n}{n} + \frac{1}{n}\right)\left(\frac{2n}{n} + \frac{1}{n}\right) \\
&= \lim_{n\to\infty} (1)\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) \\
&= \lim_{n\to\infty} (1)\left(1 + \cancelto{0}{\frac{1}{\cancelto{\infty}{n}}}\right)\left(2 + \cancelto{0}{\frac{1}{\cancelto{\infty}{n}}}\right) = (1)(1)(2) = \boxed{2}
\end{aligned}$$

8.

$$\lim_{n\to\infty}\left(\frac{8}{n^2}\right)\left(\frac{n(n+1)}{2}\right)$$

$$\lim_{n\to\infty}\left(\frac{8}{2}\right)\left(\frac{n(n+1)}{n^2}\right)=\lim_{n\to\infty}4\left(\frac{n(n+1)}{n^2}\right)$$

$$=\lim_{n\to\infty}4\left(\frac{n(n+1)}{n\cdot n}\right)\overset{\text{partner}}{=}\lim_{n\to\infty}4\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right)$$

$$=\lim_{n\to\infty}4\,(1)\left(\frac{n+1}{n}\right)\overset{\text{split}}{=}\lim_{n\to\infty}4\,(1)\left(\frac{n}{n}+\frac{1}{n}\right)$$

$$=\lim_{n\to\infty}4\,(1)\left(1+\frac{1}{n}\right)=\lim_{n\to\infty}4\,(1)\left(1+\cancelto{0}{\frac{1}{\cancelto{\infty}{n}}}\right)$$

$$=4(1)(1)=\boxed{4}$$

9.

$$\lim_{n\to\infty}\left(\frac{15}{n^2}\right)\left(\frac{n(n+1)}{2}\right)$$

$$\lim_{n\to\infty}\left(\frac{15}{2}\right)\left(\frac{n(n+1)}{n^2}\right)=\lim_{n\to\infty}\left(\frac{15}{2}\right)\left(\frac{n(n+1)}{n^2}\right)$$

$$=\lim_{n\to\infty}\left(\frac{15}{2}\right)\left(\frac{n(n+1)}{n\cdot n}\right)\overset{\text{partner}}{=}\lim_{n\to\infty}\left(\frac{15}{2}\right)\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right)$$

$$=\lim_{n\to\infty}\left(\frac{15}{2}\right)(1)\left(\frac{n+1}{n}\right)\overset{\text{split}}{=}\lim_{n\to\infty}\left(\frac{15}{2}\right)(1)\left(\frac{n}{n}+\frac{1}{n}\right)$$

$$=\lim_{n\to\infty}\left(\frac{15}{2}\right)(1)\left(1+\frac{1}{n}\right)=\lim_{n\to\infty}\left(\frac{15}{2}\right)(1)\left(1+\cancelto{0}{\frac{1}{\cancelto{\infty}{n}}}\right)$$

$$=\left(\frac{15}{2}\right)(1)(1)=\boxed{\frac{15}{2}}$$

10.

$$\lim_{n\to\infty}\left(\frac{18}{n^3}\right)\left(\frac{n(n+1)(2n+1)}{6}\right) = \lim_{n\to\infty}\left(\frac{18}{6}\right)\left(\frac{n(n+1)(2n+1)}{n^3}\right)$$

$$= \lim_{n\to\infty}(3)\left(\frac{n(n+1)(2n+1)}{n^3}\right)$$

$$= \lim_{n\to\infty}(3)\left(\frac{n(n+1)(2n+1)}{n\cdot n\cdot n}\right)$$

$$\overset{\text{partner}}{=} \lim_{n\to\infty}(3)\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)$$

$$= \lim_{n\to\infty}(3)(1)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)$$

$$\overset{\text{split}}{=} \lim_{n\to\infty}(3)(1)\left(\frac{n}{n}+\frac{1}{n}\right)\left(\frac{2n}{n}+\frac{1}{n}\right)$$

$$= \lim_{n\to\infty}(3)(1)\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

$$= \lim_{n\to\infty}(3)(1)\left(1+\cancelto{0}{\frac{1}{\cancelto{\infty}{n}}}\right)\left(2+\cancelto{0}{\frac{1}{\cancelto{\infty}{n}}}\right)$$

$$= (3)(1)(1)(2) = \boxed{6}$$

11.

$$\lim_{n\to\infty} \frac{125}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) - \frac{200}{n^2}\left(\frac{n(n+1)}{2}\right) + \left(\frac{75}{n}\right)(n)$$

$$= \lim_{n\to\infty} \left(\frac{125}{6}\right)\left(\frac{n(n+1)(2n+1)}{n^3}\right) - \left(\frac{200}{2}\right)\left(\frac{n(n+1)}{n^2}\right) + (75)\left(\frac{n}{n}\right)$$

$$= \lim_{n\to\infty} \left(\frac{125}{6}\right)\left(\frac{n(n+1)(2n+1)}{n\cdot n\cdot n}\right) - \left(\frac{200}{2}\right)\left(\frac{n(n+1)}{n\cdot n}\right) + (75)(1)$$

$$\overset{\text{partner}}{=} \lim_{n\to\infty} \left(\frac{125}{6}\right)\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) - \frac{200}{2}\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right) + 75$$

$$\overset{\text{split}}{=} \lim_{n\to\infty} \frac{125}{6}(1)\left(\frac{n}{n}+\frac{1}{n}\right)\left(\frac{2n}{n}+\frac{1}{n}\right) - (100)(1)\left(1+\frac{1}{n}\right) + 75$$

$$= \lim_{n\to\infty} \frac{125}{6}(1)\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) - (100)(1)\left(1+\frac{1}{n}\right) + 75$$

$$= \lim_{n\to\infty} \frac{125}{6}(1)\left(1+\frac{1}{n}\overset{0}{\nearrow}_{\infty}\right)\left(2+\frac{1}{n}\overset{0}{\nearrow}_{\infty}\right) - (100)(1)\left(1+\frac{1}{n}\overset{0}{\nearrow}_{\infty}\right) + 75$$

$$= \frac{125}{6}(1)(1)(2) - (100)(1)(1) + 75$$

$$= \frac{125}{3} - 100 + 75$$

$$= \frac{125}{3} - 100 + 75$$

$$= \frac{125}{3} - 25$$

$$= \frac{125}{3} - \frac{75}{3}$$

$$= \boxed{\frac{50}{3}}$$

**Summation Algebra Rules** $\displaystyle\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \ldots + a_n$

For example: $\displaystyle\sum_{i=1}^{n} i = 1 + 2 + 3 + 4 + \ldots + n$

Another one: $\displaystyle\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2 = 1 + 4 + 9 + 16 + \ldots + n^2$

We will learn more about these formulas soon.

**Specific Constant Rule for summing 1, $n$ times**

$$\sum_{i=1}^{n} 1 = \underbrace{1 + 1 + 1 + \ldots + 1}_{\text{n copies}} = n$$

**Sum/Difference Rule**

$$\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$$

**Constant Multiple Rule**

$$\sum_{i=1}^{n} \text{constant} \cdot a_i = \text{constant} \sum_{i=1}^{n} a_i$$

**Constant Rule**

$$\sum_{i=1}^{n} \text{constant} = \text{constant} \sum_{i=1}^{n} 1 = \text{constant} \cancelto{n}{\sum_{i=1}^{n}} 1 = \text{constant} \cdot n$$

1. Simplify $\displaystyle\sum_{i=1}^{n} 6 = \underbrace{6 + 6 + 6 + \ldots + 6}_{\text{n copies}} = 6n$

OR look at it in a different way, by pulling out the constant 6 from the summation notation

$$\sum_{i=1}^{n} 6 = 6 \cancelto{n}{\sum_{i=1}^{n}} 1 = 6n$$

2. Simplify $\displaystyle\sum_{i=1}^{n} (-3) = -3 \cancelto{n}{\sum_{i=1}^{n}} 1 = -3n$

3. Simplify $\displaystyle\sum_{i=1}^{n} 7 = 7 \cancelto{n}{\sum_{i=1}^{n}} 1 = 7n$

4. Simplify.

$$\sum_{i=1}^{n}\left(\frac{6i}{n}-5\right)\cdot\left(\frac{6}{n}\right) = \sum_{i=1}^{n}\left(\frac{36i}{n^2}-\frac{30}{n}\right)$$

$$\overset{\text{split}}{=} \sum_{i=1}^{n}\frac{36i}{n^2} - \sum_{i=1}^{n}\frac{30}{n}$$

$$= \frac{36}{n^2}\sum_{i=1}^{n} i - \frac{30}{n}\cancelto{n}{\sum_{i=1}^{n} 1}$$

$$= \frac{36}{n^2}\sum_{i=1}^{n} i - \left(\frac{30}{\cancel{n}}\right)\cancel{n}$$

$$= \frac{36}{n^2}\sum_{i=1}^{n} i - 30$$

5. Simplify.

$$\sum_{i=1}^{n}\left(1+\frac{3i}{n}\right)^2\cdot\left(\frac{3}{n}\right) = \sum_{i=1}^{n}\left(1+\frac{3i}{n}\right)\left(1+\frac{3i}{n}\right)\cdot\left(\frac{3}{n}\right)$$

$$= \sum_{i=1}^{n}\left(1+\frac{6i}{n}+\frac{9i^2}{n^2}\right)\cdot\left(\frac{3}{n}\right)$$

$$= \sum_{i=1}^{n}\left(\frac{3}{n}+\frac{18i}{n^2}+\frac{27i^2}{n^3}\right)$$

$$\overset{\text{split}}{=} \sum_{i=1}^{n}\frac{3}{n} + \sum_{i=1}^{n}\frac{18i}{n^2} + \sum_{i=1}^{n}\frac{27i^2}{n^3}$$

$$= \frac{3}{n}\cancelto{n}{\sum_{i=1}^{n} 1} + \frac{18}{n^2}\sum_{i=1}^{n} i + \frac{27}{n^3}\sum_{i=1}^{n} i^2$$

$$= \left(\frac{3}{\cancel{n}}\right)\cancel{n} + \frac{18}{n^2}\sum_{i=1}^{n} i + \frac{27}{n^3}\sum_{i=1}^{n} i^2$$

$$= 3 + \frac{18}{n^2}\sum_{i=1}^{n} i + \frac{27}{n^3}\sum_{i=1}^{n} i^2$$

NOTE: Soon, we will learn how to manage the summation formulas $\sum_{i=1}^{n} i$ and $\sum_{i=1}^{n} i^2$.