Tips for Physics Falling Bodies Problems

Equations of Vertical Motion

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Define $s(t)$ as the position of an object at time t.

Define $v(t)$ as the velocity at time t, the rate of change of position with respect to time.

Define $a(t)$ as the acceleration at time t, the rate of change of velocity with respect to time.

That means $v(t) = s'(t) = \frac{ds}{dt}$ is the **first** derivative of the position function.

That means $a(t) = v'(t) = \frac{dv}{dt} = s''(t)$ is the $\begin{cases} \textbf{first} \text{ derivative of the velocity function} \\ \textbf{second} \text{ derivative of the position function} \end{cases}$

$$\overrightarrow{\text{DIFFERENTIATE}}$$

$$s(t) \overset{\frac{d}{dt}}{\curvearrowright} v(t) \overset{\frac{d}{dt}}{\curvearrowright} a(t)$$

$$\overleftarrow{\int \ldots\, dt} \qquad \overleftarrow{\int dt}$$

$$\overleftarrow{ANTIDIFFERENTIATE}$$

Vertical Motion: Falling Body Problems

We will consider Vertical Motion here. On the Vertical Axis,

$$\textbf{FIX} \begin{cases} \uparrow^{+} \text{ the upwards direction as measurement in the positive + position} \\ \downarrow_{-} \text{ the downwards direction as the measurement in the negative } - \text{ position} \end{cases}$$

We fix the Accceleration due to gravity as $a(t) = -32$ feet per second squared. Then we use Antidifferentiation to solve Initial-Valued Differential Equations to find the Veloctiy and Position formulas, as derived in class.

$a(t)$	$= -32$	feet per second^2
$v(t)$	$= -32t + v(0)$ $= -32t + v_0$	feet per second
$s(t)$	$= -16t^2 + v(0)t + s(0)$ $= -16t^2 + v_0 t + s_0$	feet

Note: $\begin{cases} v_0 = v(0) & \text{is called the Initial Velocity, meaning the Velocity at time } t = 0 \\ s_0 = s(0) & \text{is called the Initial Position, meaning the Position at time } t = 0 \end{cases}$

Key Information

We typically fix the ground as Position $s(t) = 0$. There are three types of motion for an object moving vertically from a given initial height, which may even be the ground. The Object can be

- Dropped down from a fixed starting height (from rest) with no imposed initial velocity
  $\hookrightarrow v_0 = 0$
- Thrown straight Up with a given initial + velocity
  $\hookrightarrow v_0 =$ positive
- Thrown straight Down with a given initial − velocity
  $\hookrightarrow v_0 =$ negative

Recall, for functions,

$$\begin{cases} f'(x) > 0 \text{ positive} \implies f(x) \text{ is increasing} \nearrow \\ f'(x) < 0 \text{ negative} \implies f(x) \text{ is decreasing} \searrow \end{cases}$$

In this setting, these size arguments translate as follows:

$$\begin{cases} \text{Velocity } v(t) = s'(t) > 0 \text{ positive} \implies \text{Position } s(t) \text{ is increasing} \uparrow \\ \text{Velocity } v(t) = s'(t) < 0 \text{ negative} \implies \text{Position } s(t) \text{ is decreasing} \downarrow \end{cases}$$

That is, if an object is thrown UP, then the Initial Velocity is **positive** because its position is increasing (moving further in the positive direction).

OR if an onject is thrown DOWN, with a given initial force, then the Initital Velocity is **negative** because its position is decreasing (moving further in the negative direction).

Finally, when the object hits the ground, there are two key features.

• First, typically the position at *impact* is set as $s(t) = 0$. Read the problem otherwise...

• Second, the velocity at *impact* may be computed as a value, but we know v_{impact} is **negative** because the object *falling*, moving DOWN, meaning the position is more in the negative direction on the fixed vertical axis.

Practice: $\begin{cases} \text{if the position is decreasing, then its derivative (velocity) is negative} \\ \text{if the position is increasing, then its derivative (velocity) is positive} \end{cases}$

Key Types of Computations:

1. If given an input time value, then you can often **plug in** the given time t into either $s(t), v(t)$ or $a(t)$ and solve for the output value of interest.

2. If given an output function value for $s(t), v(t)$ or $a(t)$, then you can often **set** the function equal to the given output and solve for the input t. This is typically harder to do, and might involve algebra (including the Quadratic Formula).

Some Typical Questions:

- When is the Maximum Height reached? That is, t_{max} which occurs when $v(t) = 0$.

$\hookrightarrow$ Solve the derived velocity $v(t) = 0$ for time t_{max}.

- What is the Maximum Height? That is, s_{max} which occurs when $v(t) = 0$ at time t_{max}.

$\hookrightarrow$ Solve $v(t) = 0$ for time t_{max} (as described just above here) and plug that into position $s(t)$ to find the Maximum position $s_{max} = s(t_{max})$

- What is the Initial Velocity? That is, $v_0 = v(0)$.

$\hookrightarrow$ Use the derived equation for velocity and some other given information and solve for the remaining unknown v_0 in the $v(t)$ equation.

- What is the Initial Height? That is, $s_0 = s(0)$.

$\hookrightarrow$ Use the derived equation for position and some other given information and solve for the remaining unknown s_0 in the $s(t)$ equation.

- When did the object *hit the ground*? That is, t_{impact} which occurs when $s(t) = 0$.

$\hookrightarrow$ Use the derived equation for position, and solve the equation $s(t) = 0$ for the input time t labelled t_{impact}.

Again, this may involve some factoring of a Quadratic Equation or use of the Quadratice Formula. If you do factor, go ahead and try to factor off the -16 constant in front of the lead t^2 term...

- What is the Velocity at Impact when the object *hits the ground*? That is, v_{impact} which occurs when $s(t) = 0$.

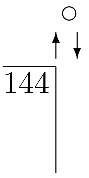
$\hookrightarrow$ Find the time of impact (as described just above here) and plug that time t_{impact} **into** the derived velocity equation. That is, solve $v_{impact} = v(t_{impact})$. It should be negative.

Final tip: Write down all the given information that is translated from the word problem. Also write down what info you're solving for, and that can often lead you to analyze a specific equation, which can make you think about which info you already know and which you can solve for.

Examples

1. A ball is thrown upward with a speed of 128 ft/sec from the edge of a cliff 144 ft above the ground. Find its height above the ground t seconds later. When does it reach its maximum height? When does it hit the ground? What is the velocity at impact?

Note $v_0 = 128\frac{\text{ft}}{\text{sec}}, s_0 = 144\text{ft}.$



$$a(t) = -32$$
$$v(t) = -32t + v_0 \Longrightarrow v(t) = -32t + 128$$
$$s(t) = -16t^2 + 128t + s_0 \Longrightarrow s(t) = -16t^2 + 128t + 144$$

Max height occurs when $v(t) = 0$. That is, $-32t + 128 = 0$ or when $t_{max} = 4$ seconds.

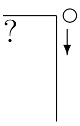
Note: if we wanted to find the Max height we would plug time $t_{max} = 4$ seconds into the position equation.

The ball hits the ground when $s(t) = -16t^2 + 128t + 144 = 0$ or when $-16(t^2 - 8t - 9) = 0$ so that $-16(t - 9)(t + 1) = 0$. Then $t = 9$ or $t = -1$. We will ignore the negative time here. The ball hits the ground after 9 seconds. That is, $t_{impact} = 9$ seconds.

The velocity at impact is given by $v(t_{impact}) = v(9) = -32(9) + 128 = -288 + 128 = -160$ feet per second. The negative sign makes sense, since the ball is going down, with decreasing position at impact.

2. Jack throws a baseball straight downward from the top of a tall building. The initial speed of the ball is 25 feet per second. It hits the ground with a speed of 153 feet per second. How tall is the building?

Note $v_0 = -25\frac{\text{ft}}{\text{sec}}, s_0 = ?\text{ft}, v_{\text{impact}} = -153\frac{\text{ft}}{\text{sec}}$



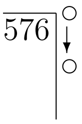
$$a(t) = -32$$
$$v(t) = -32t + v_0 \Longrightarrow v(t) = -32t - 25$$
$$s(t) = -16t^2 - 25t + s_0$$

The ball hits the ground when $v(t) = -32t - 25 = -153$ or when $32t = 153 - 25 = 128$ which is when $t_{\text{impact}} = 4$ seconds.

Finally, we solve $s(4) = 0$ for s_0. The ball hits the ground when $-16(4)^2 - 25(4) + s_0 = 0$ or when $-256 - 100 + s_0 = 0$ which is when $s_0 = 356$ feet. As a result, the building is 356 feet tall.

3. A ball is dropped from the top of the building 576 feet high. How much time passes before the ball hits the ground? What is the velocity at impact?

Note $v_0 = 0\ \frac{\text{ft}}{\text{sec}}, s_0 = 576\text{ft}.$



The ball has the following motion equations:

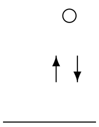
$a(t) = -32$
$v(t) = -32t + v_0 \Longrightarrow v(t) = -32t + 0 \Longrightarrow v(t) = -32t$
$s(t) = -16t^2 + v(0)t + s_0 \Longrightarrow s(t) = -16t^2 + 0t + 576 \Longrightarrow s(t) = -16t^2 + 576$

The ball hits the ground when $s(t) = -16t^2 + 576 = 0$ or when $16t^2 = 576$ which is when $t^2 = \dfrac{576}{16}$ or when $t^2 = 36$ which yields $t_{\text{impact}} = 6$ seconds.

Finally, the velocity at impact is given by $v(6) = -32(6) = -192$ feet per second.

4. Suppose that a bolt was fired vertically upward from a powerful crossbow at ground level, and that it struck the ground 48 seconds later. If air resistance may be neglected, find the initial velocity of the bolt and the maximum height it reached.

Note $v_0 = ?\frac{\text{ft}}{\text{sec}}, s_0 = 0\text{ ft}, t_{\text{impact}} = 48\text{sec}.$



$a(t) = -32$
$v(t) = -32t + v_0$
$s(t) = -16t^2 + v_0 t + s_0 \Longrightarrow s(t) = -16t^2 + v_0 t + 0 \Longrightarrow s(t) = -16t^2 + v_0 t$

First we use the impact information, $s(48) = 0$.

$s(48) = 0$
$-16(48)^2 + v_0(48) = 0 \Longrightarrow 48v_0 = 16(48)^2 \Longrightarrow v_0 = \dfrac{16(48)^2}{48} = 16(48) = 768\dfrac{\text{ft}}{\text{sec}},$

As a result $v(t) = -32t + 768$ and the max height occurs when $v(t) = 0 \Longrightarrow t_{max} = \dfrac{768}{32} = 24$ seconds. Finally the max height is $s(24) = -16(24)^2 + 768(24) = -16(576) + 18432 = -9216 + 18432 = 9216$ feet.