Math 106 Final May 15, 2023

• This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, webpages, or other aids are permitted.
• You need *not* simplify algebraically complicated answers. However, numerical answers such as $\sin\left(\frac{\pi}{6}\right)$, $4^{\frac{3}{2}}$, $e^{\ln 4}$, $\ln(e^7)$, $e^{-\ln 5}$, or $e^{3\ln 3}$ should be simplified.
• Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

1. [24 Points] Compute each of the following derivatives.

(a) $\frac{dy}{dx}$, where $y = (\ln x)^{\ln x}$. Simplify. (b) $\frac{d}{dx} \ln\left(\frac{(6-x^4)^8 \cdot e^{\tan x}}{\sqrt{9-x^2}\cdot e^{-6x}}\right)$ Do not simplify.

(c) Find the Local Max/Min **Values** for $f(x) = \frac{x^4}{e^x}$. (d) Prove $\frac{d}{dx}\ln x = \frac{1}{x}$.

2. [18 Points] Compute the following Derivative. Simplify.

(a) Consider $f(x) = \cos(7x) + \cos(6x) + \sin\left(\frac{\pi}{3}\right) + \sin(3x) + \sin(4x)$ Show that $f'\left(\frac{\pi}{6}\right) = \boxed{\frac{3}{2}}$

(b) Consider $f(x) = \cos(\ln(1+4x)) - \ln(1) - \ln(1+\cos(5x)) - e^{\tan(6x)} - \sin\left(e^{7x} - 1\right)$ Show that $f'(0) = \boxed{-13}$

3. [24 Points] Compute each of the following Integrals.

(a) $\int \frac{\sec^2(\sqrt{x}\,)}{\sqrt{x}}\, dx$ (b) $\int \frac{x^3}{(9-x^2)^2}\, dx$

(c) $\int \left(e^{3x} + \frac{1}{e^{2x}}\right)\left(e^x + \frac{1}{e^{5x}}\right) dx$ (d) $\int \frac{\sin x}{e^{\cos x}}\, dx$

4. [40 Points] Compute the following Definite Integrals. Match the given answer.

(a) $\int_{\ln 2}^{\ln 4} \frac{1}{e^{2x}\left(1 - e^{-2x}\right)}\, dx = \boxed{\ln\left(\frac{\sqrt{5}}{2}\right)}$ (b) $\int_{e^2}^{e^7} \frac{8}{x\sqrt{2+\ln x}}\, dx = \boxed{16}$

(c) $\int_1^3 \frac{1}{x^2}\sin\left(\frac{\pi}{x}\right) dx = \boxed{\frac{3}{2\pi}}$ (d) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2 x}{\tan^3 x}\, dx = \boxed{\frac{1}{3}}$ (e) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \tan x\; dx = \boxed{\frac{1}{2}\ln 3}$

5. [18 Points] Consider $f(x) = e^{7x} - \dfrac{1}{e^{7x}} + e^{e^{e^7}} + x^e + \ln(\ln(e^7)) + \dfrac{7}{x} - \dfrac{1}{e^x} + 7e^x + \dfrac{e^x}{e^{7x}}$

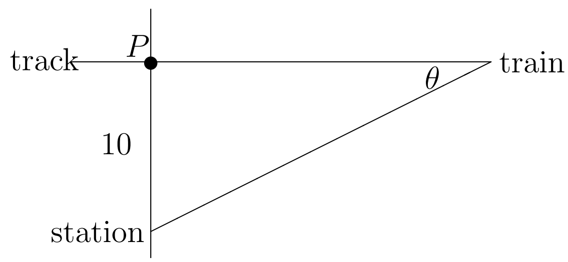
(a) Compute the **Derivative** $f'(x)$. (b) Compute the **Antiderivative** $\displaystyle\int f(x)\, dx$.

6. [16 Points] Show that $\displaystyle\int_{-1}^{2} 2 - 2x - x^2\, dx = \boxed{0}$ using **TWO** different methods:
(a) Fundamental Theorem of Calculus (b) Limit Definition of the Definite Integral

7. [12 Points] Consider a point P on a train track. Suppose a train depot station is 10 feet directly south from this point P. The train is travelling East at 6 feet per second. Consider the angle θ as shown in the diagram. How fast is this angle θ changing when the distance between the train and the station is 20 feet?

• Diagram

The picture at arbitrary time t is:



8. [24 Points] (a) Consider the region bounded by $y = e^x + 1$, $y = 4$, and $x = 0$. Sketch and shade the bounded region.
(b) **COMPUTE** the **Area** bounded in part (a).
(c) **COMPUTE** the **Volume** of the three-dimensional solid obtained by rotating the region in (a) about the $\boldsymbol{x}$**-axis**. Sketch the solid, along with one of the approximating *Washers.*

(d) Consider a **different** region bounded by $y = \ln x$, $y = 0$, $x = 1$ and $x = 3$. Sketch and shade the bounded region.
(e) **Set-Up** but **DO NOT EVALUATE** the integral to compute the **Volume** of the three-dimensional solid obtained by rotating the region in (d) about the line $\boldsymbol{y = -3}$. Sketch the solid, along with one of the approximating *Washers.*

9. [10 Points] Jack throws a baseball straight upwards from the top of a building. This initial velocity of the ball is 32 feet per second. The ball hits the ground with a velocity of -64 feet per second. Draw a sketch. How tall is the building? When is the Maximum Height reached? What is the Maximum Height reached?

10. [14 Points] (a) A population of bacteria was growing exponentially. Initially there were 3 cells. After 2 hours there were 12 cells. How many cells were there after 6 hours? Simplify. When were there 999 cells? (You can leave this time answer in terms of logs)

(b) A new car costs \$10,000 with value decreasing exponentially each year. After 3 years, the value of the car is \$8,000. First find the Decay Constant k. What will the value of the car be after 9 years? Hint: $10{,}000\left(\dfrac{4}{5}\right)^3 = \$5{,}120$.