# Fundamental Theorem of Calculus

## Fundamental Theorem of Calculus, Part 1

If $f$ is continuous on $[a, b]$, then the function $g$ defined by

$$g(x) = \int_a^x f(t)\ dt \quad a \leq x \leq b$$

is continuous on$[a, b]$ and differentiable on$(a, b)$, and $g'(x) = f(x)$.
That is,

$$g'(x) = \frac{d}{dx} \int_a^x f(t)\ dt = f(x)$$

Think: if you have a Definite Integral with a constant $a$ as a Lower Limit of Integration, and a variable (here as $x$) as the Upper Limit, then the Derivative of the *variable* Integral is equal to the Integrand $f$ evaluated simply at $x$. More simply put, you get back the function inside the integral and plug in the single variable $x$. The order of the limits is important here.

Example: If $g(x) = \displaystyle\int_x^3 \sqrt{5 \cos t}\ dt$,

then $g'(x) = \dfrac{d}{dx} \displaystyle\int_x^3 \sqrt{5 + \cos t}\ dt = \dfrac{d}{dx}\left( - \displaystyle\int_3^x \sqrt{5 + \cos t}\ dt \right) = \boxed{-\sqrt{5 + \cos x}}$

## Fundamental Theorem of Calculus, Part 2

If $f$ is continuous on $[a, b]$, then

$$\int_a^b f(x)\ dx = F(x)\Big|_a^b = F(b) - F(a)$$

where $F$ is any Antiderivative of $f$, that is, a function $F$ such that $F' = f$.

Example:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 6 \cos x\ dx = 6 \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos x\ dx = 6 \sin x \Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} = 6 \left( \sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{6}\right) \right)$$

$$= 6\left(1 - \frac{1}{2}\right) = 6\left(\frac{1}{2}\right) = \boxed{3}$$

**Properties of the Definite Integral**

1. Empty Area: $\displaystyle\int_a^a f(x)\ dx = 0$

2. Limits Order: $\displaystyle\int_b^a f(x)\ dx = -\int_a^b f(x)\ dx$

3. Constant Rule: $\displaystyle\int_a^b constant\ dx = constant \cdot (b-a)$

4. Constant Multiple Rule: $\displaystyle\int_a^b cf(x)\ dx = c\int_a^b f(x)\ dx$ where $c$ is a constant

5. Summation Rule: $\displaystyle\int_a^b f(x) + g(x)\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx$

6. Difference Rule: $\displaystyle\int_a^b f(x) - g(x)\ dx = \int_a^b f(x)\ dx - \int_a^b g(x)\ dx$

7. Split Area Rule: $\displaystyle\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$ for $c$ with $a < c < b$

**IMPORTANT:** There is **NO** immediate Integration Rule for Products or Quotients.

• You can use Algebra to simplify the integrand into simpler Power Rule or light Trig pieces.

OR

• You can use $u$-substitution for special combos where a certain (usually nested) chunk of the integrand can temporarily be hidden, as say $u$, *and* the derivative piece $du$ is also found in the same problem.