Extra Riemann Sum, Definite Integral Example

Evaluate $\int_{-1}^{2} x^2 - 3x + 6 \, dx$ using Riemann Sums and the limit definition of the definite integral.

Here $f(x) = x^2 - 3x + 6, \quad a = -1, \quad b = 2, \quad \Delta x = \frac{2-(-1)}{n} = \frac{b-a}{n} = \frac{3}{n}$

and $x_i = a + i\Delta x = -1 + i\left(\frac{3}{n}\right) = -1 + \frac{3i}{n}$.

$$\begin{aligned}
\int_{-1}^{2} x^2 - 3x + 6 \, dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(-1 + \frac{3i}{n}\right)\left(\frac{3}{n}\right) \\
&= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left(-1 + \frac{3i}{n}\right)^2 - 3\left(-1 + \frac{3i}{n}\right) + 6 \\
&= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left(1 - \frac{6i}{n} + \frac{9i^2}{n^2} + 3 - \frac{9i}{n} + 6\right) \\
&= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left(\frac{9i^2}{n^2} - \frac{15i}{n} + 10\right) \\
&= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \frac{9i^2}{n^2} - \frac{3}{n} \sum_{i=1}^{n} \frac{15i}{n} + \frac{3}{n} \sum_{i=1}^{n} 10 \\
&= \lim_{n\to\infty} \frac{27}{n^3} \sum_{i=1}^{n} i^2 - \frac{45}{n^2} \sum_{i=1}^{n} i + \frac{30}{n} \sum_{i=1}^{n} 1 \\
&= \lim_{n\to\infty} \frac{27}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) - \frac{45}{n^2}\left(\frac{n(n+1)}{2}\right) + \frac{30}{\cancel{n}}(\cancel{n}) \\
&= \lim_{n\to\infty} \frac{27}{6}\left(\frac{\cancel{n}}{\cancel{n}}\right)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) - \frac{45}{2}\left(\frac{\cancel{n}}{\cancel{n}}\right)\left(\frac{n+1}{n}\right) + 30 \\
&= \lim_{n\to\infty} \frac{27}{6}(1)\left(1 + \cancelto{0}{\frac{1}{n}}\right)\left(2 + \cancelto{0}{\frac{1}{n}}\right) - \frac{45}{2}(1)\left(1 + \cancelto{0}{\frac{1}{n}}\right) + 30 \\
&= \frac{27}{6}(1)(1)(2) - \frac{45}{2}(1)(1) + 30 = \frac{54}{6} - \frac{45}{2} + 30 \\
&= \frac{54}{6} - \frac{135}{6} + \frac{180}{6} = \frac{99}{6} = \boxed{\frac{33}{2}}
\end{aligned}$$