Math 105, Fall 2013

## Worksheet 12, Tuesday, December 10, 2013, Answer Key

### Max-Min Problems

(Remember to state the domain of the function you are computing extreme values for.)

1. Let $R$ be the region between $y = 9 - x^2$ and the $x$-axis. Find the area of the largest rectangle that can be inscribed in the region $R$. Two vertices of the rectangle lie on the $x$-axis. Its other two vertices above the $x$-axis lie on the parabola $y = 9 - x^2$.

- Diagram: See me for a diagram.

- Variables:
  Let $x = x$–coordinate of a point on the parabola, upper right corner of rect.
  Let $y = y$–coordinate of a point on the parabola, upper right corner of rect.
  Let $A$=Area of inscribed rectangle.

- Equations:

$y = 9 - x^2 \longleftarrow$ Fixed

$A = 2xy = 2x(9 - x^2) = 18x - 2x^3 \longleftarrow$ Maximize

The (common-sense-bounds)domain of $A$ is $\{x : 0 \le x \le 3\}$.

- Maximize:

Next $A' = 18 - 6x^2$. Setting $A' = 0$ we solve for $x^2 = 3$ or $x = \sqrt{3}$. (We take the positive square root here because we're talking distance.)

Sign-testing the critical number does indeed yield a maximum for the area function.

| | | | |
|---|---|---|---|
| $A'$ | $\oplus$ | | $\ominus$ |
| $A$ | $\nearrow$ | $\sqrt{3}$ | $\searrow$ |
| | | MAX | |

Since $x = \sqrt{3}$ then $y = 9 - (\sqrt{3})^2 = 6$. Then $A = 2xy = 2\sqrt{3}(6) = 12\sqrt{3}$

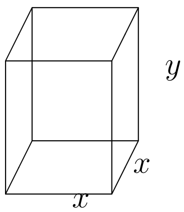
OR use $A = 18x - 2x^3 = 18(\sqrt{3}) - 2(\sqrt{3})^3 = 18\sqrt{3} - 6\sqrt{3} = 12\sqrt{3}$.

- Answer:

As a result, the largest area that occurs is $A = 12\sqrt{3}$ square units.

2. A rectangular box with square base cost \$ 2 per square foot for the bottom and \$ 1 per square foot for the top and sides. Find the box of largest volume which can be built for \$36.

- Diagram:




- Variables:

Let $x$ =length of side on base of box.
Let $y$ =height of box.
Let Cost=Cost for amount of material (surface area).
Let $V$ =volume of box.

- Equations:

Then the Cost of materials, which is fixed is given as

$$\begin{aligned}\text{Cost} &= \text{cost of base} + \text{cost of top} + \text{cost of 4 sides} \\ &= x^2(\$2) + x^2(\$1) + 4xy(\$1) \\ &= 3x^2 + 4xy = 36 \ \longleftarrow \text{Fixed} \\ &\Longrightarrow y = \frac{36 - 3x^2}{4x}\end{aligned}$$

$$V = x^2 y = x^2\left(\frac{36 - 3x^2}{4x}\right) = 9x - \frac{3}{4}x^3 \longleftarrow \text{Maximize}$$

(Try to simplify before you differentiate.)

The (common-sense-bounds)domain of $V$ is $\{x : 0 < x \leq \sqrt{12}\}$.

- Maximize:

Next $V' = 9 - \frac{9}{4}x^2$. Setting $V' = 0$ we solve for $x = 2$ as the critical number.

Sign-testing the critical number does indeed yield a maximum for the volume function.

V′ ⊕ ⊖
V ↗ 2 ↘
MAX

Since $x = 2$ then $y = \frac{36 - 12}{8} = 3$.

- Answer:

As a result, the box of largest volume will measure **2x2x3**, each in feet.

**Review**–early limits and derivatives

3. Evaluate each of the following limits. Please **justify** your answers. Be clear if the limit equals a value, or $+\infty$ or $-\infty$, or Does Not Exist.

(a) $\displaystyle \lim_{x\to-2} \frac{x^2+3x+2}{x^2+x-2} = \lim_{x\to-2} \frac{(x+2)(x+1)}{(x+2)(x-1)} = \lim_{x\to-2} \frac{x+1}{x-1} = \frac{-1}{-3} = \boxed{\frac{1}{3}}$

(b) $\displaystyle \lim_{x\to1^-} \frac{g(x+1)-2x-5}{(x-1)^2}$, where $g(x) = x^2+3$.

$$\lim_{x\to1^-} \frac{g(x+1)-2x-5}{(x-1)^2} = \lim_{x\to1^-} \frac{(x+1)^2+3-2x-5}{(x-1)^2}$$

$$= \lim_{x\to1^-} \frac{x^2+2x+1+3-2x-5}{(x-1)^2} = \lim_{x\to1^-} \frac{x^2-1}{(x-1)^2}$$

$$= \lim_{x\to1^-} \frac{(x-1)(x+1)}{(x-1)^2} = \lim_{x\to1^-} \frac{x+1}{x-1} = \frac{2}{0^-} = \boxed{-\infty}$$

(c) $\displaystyle \lim_{x\to\infty} \frac{3-2x^2}{3x^2+5x} = \lim_{x\to\infty} \frac{3-2x^2}{3x^2+5x} \cdot \frac{\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x^2}\right)} = \lim_{x\to\infty} \frac{\frac{3}{x^2} - \frac{2x^2}{x^2}}{\frac{3x^2}{x^2} + \frac{5x}{x^2}}$

$$= \lim_{x\to\infty} \frac{\frac{3}{x^2} - 2}{3 + \frac{5}{x}} = \boxed{-\frac{2}{3}}$$

(d) $\displaystyle \lim_{x\to5} \frac{25-x^2}{\sqrt{x+4}-3} = \lim_{x\to5} \frac{25-x^2}{\sqrt{x+4}-3} \cdot \frac{\sqrt{x+4}+3}{\sqrt{x+4}+3}$

$$= \lim_{x\to5} \frac{(25-x^2)(\sqrt{x+4}+3)}{(x+4)-9} = \lim_{x\to5} \frac{(5-x)(5+x)(\sqrt{x+4}+3)}{x-5}$$

$$= \lim_{x\to5} \frac{-(x-5)(5+x)(\sqrt{x+4}+3)}{x-5} = \lim_{x\to5} -(5+x)(\sqrt{x+4}+3)$$

$$= -10(\sqrt{9}+3) = -10(3+3) = -10(6) = \boxed{-60}$$

(e) $\displaystyle \lim_{x\to7} \frac{|7-x|}{x^2-x-42}$

$\boxed{\text{DNE}}$ since RHL$\neq$LHL. see below.

Note: $|7-x| = \begin{cases} 7-x & \text{if } 7-x \geq 0, \text{ that is } x \leq 7 \longleftarrow \text{LHL} \\ -(7-x) & \text{if } 7-x < 0, \text{ that is } x > 7 \longleftarrow \text{RHL} \end{cases}$

RHL: $\lim_{x\to 7+} \dfrac{|7-x|}{x^2-x-42} = \lim_{x\to 7+} \dfrac{-(7-x)}{x^2-x-42} = \lim_{x\to 7+} \dfrac{x-7}{(x-7)(x+6)}$

$= \lim_{x\to 7+} \dfrac{1}{x+6} = \dfrac{1}{13}$

LHL: $\lim_{x\to 7-} \dfrac{|7-x|}{x^2-x-42} = \lim_{x\to 7+} \dfrac{7-x}{x^2-x-42} = \lim_{x\to 7+} \dfrac{-(x-7)}{(x-7)(x+6)}$

$= \lim_{x\to 7+} \dfrac{-1}{x+6} = -\dfrac{1}{13}$

(f) $\lim_{x\to -5} \dfrac{\dfrac{5}{x} - \dfrac{1}{x+4}}{x+5} = \lim_{x\to -5} \dfrac{\dfrac{5(x+4)-x}{x(x+4)}}{x+5} = \lim_{x\to -5} \dfrac{5x+20-x}{x(x+4)(x+5)}$

$= \lim_{x\to -5} \dfrac{4x+20}{x(x+4)(x+5)} = \lim_{x\to -5} \dfrac{4(x+5)}{x(x+4)(x+5)} = \lim_{x\to -5} \dfrac{4}{x(x+4)}$

$= \dfrac{4}{(-5)(-5+4)} = \boxed{\dfrac{4}{5}}$

4. Compute the derivative for each of the following functions. Do not simplify your answers here.

(a) $f(x) = \dfrac{\sqrt{x^3 - x^{-8}}}{(x^2+5)^4}$

$$f'(x) = \boxed{\dfrac{(x^2+5)^4\left(\dfrac{1}{2\sqrt{x^3-x^{-8}}}\right)(3x^2+8x^{-9}) - \sqrt{x^3-x^{-8}}(4)(x^2+5)^3(2x)}{(x^2+5)^8}}$$

(b) $f(x) = \left(\dfrac{x^{\frac{3}{2}} + x^{\frac{2}{3}}}{2\sqrt{x}-3}\right)^4$

$$f'(x) = \boxed{4\left(\dfrac{x^{\frac{3}{2}} + x^{\frac{2}{3}}}{2\sqrt{x}-3}\right)^3 \left(\dfrac{(2\sqrt{x}-3)\left(\left(\dfrac{3}{2}\right)x^{\frac{1}{2}} + \left(\dfrac{2}{3}\right)x^{-\frac{1}{3}}\right) - \left(x^{\frac{3}{2}} + x^{\frac{2}{3}}\right)\left(\dfrac{1}{2\sqrt{x}}\right)}{(2\sqrt{x}-3)^2}\right)}$$

(c) $f(x) = \left(\frac{1}{x^3} - \frac{1}{x^7}\right)\sqrt{\frac{7}{x} - \frac{3x}{7}}$

$$f'(x) = \boxed{\left(\frac{1}{x^3} - \frac{1}{x^7}\right)\left(\frac{1}{2\sqrt{\frac{7}{x} - \frac{3}{7x}}}\right)\left(-\frac{7}{x^2} - \frac{3}{7}\right) + \sqrt{\frac{7}{x} - \frac{3x}{7}}\left(-\frac{3}{x^4} + \frac{7}{x^8}\right)}$$

5. Let $f(x) = \dfrac{x+1}{x+2}$. Calculate $f'(x)$, using the **limit definition** of the derivative.

Check your answer using the Quotient Rule.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{\dfrac{(x+h)+1}{(x+h)+2} - \dfrac{x+1}{x+2}}{h}$$

$$= \lim_{h\to 0}\frac{\dfrac{(x+h+1)(x+2) - (x+1)(x+h+2)}{(x+h+2)(x+2)}}{h}$$

$$= \lim_{h\to 0}\frac{(x^2 + xh + x + 2x + 2h + 2) - (x^2 + xh + 2x + x + h + 2)}{h(x+h+2)(x+2)}$$

$$= \lim_{h\to 0}\frac{(x^2 + xh + 3x + 2h + 2) - (x^2 + xh + 3x + h + 2)}{h(x+h+2)(x+2)}$$

$$= \lim_{h\to 0}\frac{x^2 + xh + 3x + 2h + 2 - x^2 - xh - 3x - h - 2}{h(x+h+2)(x+2)}$$

$$= \lim_{h\to 0}\frac{h}{h(x+h+2)(x+2)}$$

$$= \lim_{h\to 0}\frac{1}{(x+h+2)(x+2)} = \boxed{\frac{1}{(x+2)^2}}$$

Double check using quotient rule:

$$f'(x) = \frac{(x+2)(1) - (x+1)(1)}{(x+2)^2} = \frac{x+2-x-1}{(x+2)^2} = \boxed{\frac{1}{(x+2)^2}}$$