Math 105, Fall 2013

Worksheet 11, Tuesday, December 3, 2013

Critical Numbers

1. Find critical numbers for the function $f(x) = x^{\frac{4}{5}}(x-4)^2$.

   First $f'(x) = x^{\frac{4}{5}}2(x-4) + (x-4)^2\left(\frac{4}{5}\right)x^{-\frac{1}{5}} = x^{\frac{4}{5}}2(x-4) + \frac{4(x-4)^2}{5x^{\frac{1}{5}}}$

   $= \left(\frac{5x^{\frac{1}{5}}}{5x^{\frac{1}{5}}}\right)x^{\frac{4}{5}}2(x-4) + \frac{4(x-4)^2}{5x^{\frac{1}{5}}} = \frac{10x(x-4)}{5x^{\frac{1}{5}}} + \frac{4(x-4)^2}{5x^{\frac{1}{5}}}$

   $= \frac{10x(x-4)+4(x-4)^2}{5x^{\frac{1}{5}}} = \frac{2(x-4)[5x+2(x-4)]}{5x^{\frac{1}{5}}} = \frac{2(x-4)[7x-8]}{5x^{\frac{1}{5}}} = 0$

   when the numerator equals 0, which is when $x-4=0$ or $7x-8=0$, which happens when $x=4$ or $x=\frac{8}{7}$.

   Secondly the derivative is underfined when the denominator equals 0 here, when $x=0$.

   Finally the critical numbers are $\boxed{x=4,\ x=\frac{8}{7}}$ and $\boxed{x=0}$

2. Find critical numbers for the function $f(x) = x^{\frac{1}{3}} - x^{-\frac{2}{3}}$.

   First $f'(x) = \frac{1}{3}x^{-\frac{2}{3}} + \frac{2}{3}x^{-\frac{5}{3}} = \frac{1}{3x^{\frac{2}{3}}} + \frac{2}{3x^{\frac{5}{3}}} = \left(\frac{x}{x}\right)\frac{1}{3x^{\frac{2}{3}}} + \frac{2}{3x^{\frac{5}{3}}} = \frac{x}{3x^{\frac{5}{3}}} + \frac{2}{3x^{\frac{5}{3}}} = \frac{x+2}{3x^{\frac{5}{3}}} = 0$

   when the numerator equals 0, which is when $x+2=0$ or when $x=-2$.

   Secondly the derivative is underfined when the denominator equals 0 here, when $x=0$. However, $x=0$ was no in the domain of the original function. Hence it is not technically a critical number.

   Finally the critical number is $\boxed{x=-2}$

Absolute Extreme Values

3. Find the absolute maximum and absolute minimum values of

$$h(x) = 3(x+1)^{\frac{2}{3}} - x \text{ on } [0, 26].$$

$$h'(x) = 2(x+1)^{-\frac{1}{3}} - 1 = \frac{2}{(x+1)^{\frac{1}{3}}} - \frac{(x+1)^{\frac{1}{3}}}{(x+1)^{\frac{1}{3}}} = \frac{2-(x+1)^{\frac{1}{3}}}{(x+1)^{\frac{1}{3}}} = 0$$

when $2-(x+1)^{\frac{1}{3}} = 0$ or when $(x+1)^{\frac{1}{3}} = 2$ which happens when $x+1=8$ or finally when $x=7$. Next, $h'(x)$ is undefined when $x=-1$ which was in the domain of the original function h. Our critical numbers are $x = 7$ and $x = -1$, but $x = -1$ is outside of our interval of interest.

Applying the closed interval method:

$h(7) = 3(8)^{\frac{2}{3}} - 7 = 3\left((8)^{\frac{1}{3}}\right)^2 - 7 = 3\,(2)^2 - 7 = 3(4) - 7 = 12 - 7 = \boxed{5} \longleftarrow$ Absolute Maximum Value

$h(0) = 3(0+1)^{\frac{2}{3}} - 0 = 3 - 0 = 3$

$h(26) = 3(26+1)^{\frac{2}{3}} - 26 = 3\left((27)^{\frac{1}{3}}\right)^2 - 26 = 3\,(3)^2 - 26 = 3(9) - 26 = 27 - 26 = \boxed{1} \longleftarrow$ Absolute Maximum Value

So the absolute maximum value is 5 (attained at $x = 7$), and the absolute minimum value is 1 (attained at $x = 26$) on this closed interval $[0, 26]$.

4. Find the absolute maximum and absolute minimum values of

$$f(x) = (x-1)^2(2x+10)^2 \text{ on } [-3, 2].$$

$f'(x) = (x-1)^2 \cdot 2(2x+10)(2) + (2x+10)^2 \cdot 2(x-1) = 2(x-1)(2x+10)[2(x-1) + (2x+10)] = 2(x-1)(2x+10)[4x+8]$. Here f' is always defined. Also, $f'(x) = 0$ happens only when $x = 1$, $x = -5$, and $x = -2$ (our critical numbers). Here $x = -5$ is outside of our interval of interest. Applying the closed interval method:

$f(1) = \boxed{0} \longleftarrow$ Absolute Minimum Value

$f(-2) = (-3)^2(6)^2 = 9(36) = \boxed{324} \longleftarrow$ Absolute Maximum Value

$f(-3) = (-4)^2(4)^2 = (16)(16) = 256$

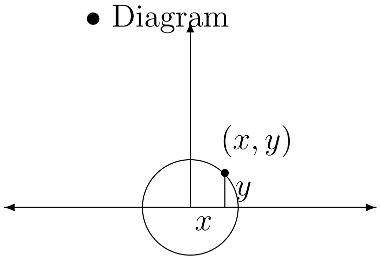
$f(2) = (1)^2(14)^2 = 196.$

So the absolute maximum value is 324 (attained at $x = -2$), and the absolute minimum value is 0 (attained at $x = 1$) on this closed interval.

Related Rates

5. A point moves around the circle $x^2 + y^2 = 9$. When the point is at $(-\sqrt{3}, \sqrt{6})$, its x-coordinate is increasing at the rate of 20 units per second. How fast is its y-coordinate changing at this instant?

- Diagram



- Variables

Let $x =$ the x-coord. of the point at time t
Let $y =$ the y-coord. of the point at time t
Find $\frac{dy}{dt} = ?$ when $x = -\sqrt{3}$, $y = \sqrt{6}$
and $\frac{dx}{dt} = 20\frac{\text{units}}{\text{sec}}$

• Equation relating the variables:

Given as $x^2 + y^2 = 9$.

• Differentiate both sides w.r.t. time t.

$\frac{d}{dt}(x^2+y^2) = \frac{d}{dt}(9) \implies 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \implies x\frac{dx}{dt} + y\frac{dy}{dt} = 0$ (Related Rates!)

• Substitute Key Moment Information (now and not before now!!!):

$(-\sqrt{3})20 + \sqrt{6}\frac{dy}{dt} = 0$

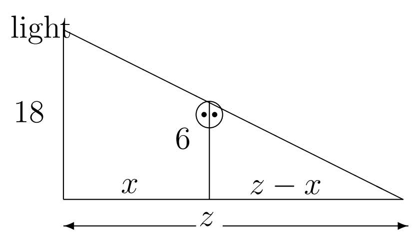
• Solve for the desired quantity:

$\frac{dy}{dt} = \frac{20\sqrt{3}}{\sqrt{6}}$

• Answer the question that was asked: At this moment, the y-coordinate is increasing at a rate of $\frac{20\sqrt{3}}{\sqrt{6}}$ units every second.

6. A 6 foot tall man walks with a speed of 8 feet per second away from a street light that is atop an 18 foot pole. How fast is the top of shadow moving long the ground when he is 100 feet from the light pole?

• Diagram



• Variables

Let x = man's distance from pole at time t

Let z = distance from tip of shadow to pole at time t

Find $\frac{dz}{dt}$ =? when $x = 100$ feet

and $\frac{dx}{dt} = 8\frac{\text{ft}}{\text{sec}}$ He's fast!

• Equation relating the variables:

Via similar triangles, we must have

$$\frac{z}{18} = \frac{z-x}{6} \implies 6z = 18z - 18x \implies 18x = 12z \implies 3x = 2z$$

• Differentiate both sides w.r.t. time t.

$\frac{d}{dt}(3x) = \frac{d}{dt}(2z) \implies 3\frac{dx}{dt} = 2\frac{dz}{dt}$(Related Rates!)

• Substitute Key Moment Information (now and not before now!!!):

$3(8) = 2\dfrac{dz}{dt}$

• Solve for the desired quantity:

$\dfrac{dz}{dt} = 12\dfrac{\text{ft}}{\text{sec}}$

• Answer the question that was asked: The tip of his shadow is moving along the ground at a rate of 12 feet every second (fast). Note that the rate is independent of the man's distance from the pole...

Limits at Infinity

7. Compute each of the following limits at infinity. Please show your work.

(a) $$\lim_{x\to\infty} \frac{x^7 - 4x + 2013}{x^6 + 3x^2 - 8} = \lim_{x\to\infty} \frac{x^7 - 4x + 2013}{x^6 + 3x^2 - 8} \cdot \frac{\left(\frac{1}{x^6}\right)}{\left(\frac{1}{x^6}\right)}$$

$$= \lim_{x\to\infty} \frac{x - \frac{4}{x^5} + \frac{2013}{x^6}}{1 + \frac{3}{x^4} - \frac{8}{x^6}} = \boxed{\infty}$$

(b) $$\lim_{x\to-\infty} \frac{7x^3 - 8x + 19}{4x^3 + 2x^2 - 3x + 5} = \lim_{x\to-\infty} \frac{7x^3 - 8x + 19}{4x^3 + 2x^2 - 3x + 5} \cdot \frac{\left(\frac{1}{x^3}\right)}{\left(\frac{1}{x^3}\right)}$$

$$= \lim_{x\to-\infty} \frac{7 - \frac{8}{x^2} + \frac{19}{x^3}}{4 + \frac{2}{x} - \frac{3}{x^2} + \frac{5}{x^3}} = \boxed{\frac{7}{4}}$$

(c) $$\lim_{x\to\infty} \frac{1 - x^2}{8x^4 + 200} = \lim_{x\to\infty} \frac{1 - x^2}{8x^4 + 200} \cdot \frac{\left(\frac{1}{x^4}\right)}{\left(\frac{1}{x^4}\right)} = \lim_{x\to\infty} \frac{\frac{1}{x^4} - \frac{1}{x^2}}{8 + \frac{200}{x^4}} = \boxed{0}$$

Curve Sketching For the following function, discuss domain, vertical and horizontal asymptotes, intervals of increase or decrease, local extreme value(s), concavity, and inflection point(s). Then use this information to present a detailed and labelled sketch of the curve.

8. $f(x) = \dfrac{2x^3 + 45x^2 + 315x + 600}{x^3}$. Take my word for it that (you do NOT have to compute these) $f'(x) = \dfrac{-45(x+4)(x+10)}{x^4}$ and $f''(x) = \dfrac{90(x+5)(x+16)}{x^5}$.

Hint: You might need the following values:

$f(-4) = \dfrac{17}{16}$, $\quad f(-10) = \dfrac{1}{20}$, $\ f(-16) = \dfrac{139}{512}$, $\quad$ and finally $f(-5) = \dfrac{4}{5}$.

- Domain: $f(x)$ has domain $\{x|x \neq 0\}$
- VA: Vertical asymptotes at $x = 0$.
- HA: Horizontal asymptote at $y = 2$ for this f since

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{2x^3+45x^2+315x+600}{x^3} = \lim_{x\to\pm\infty} \frac{2x^3+45x^2+315x+600}{x^3} \cdot \frac{\left(\dfrac{1}{x^3}\right)}{\left(\dfrac{1}{x^3}\right)}$$

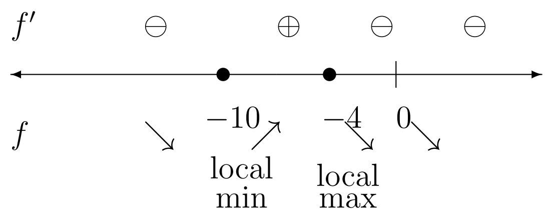
$$= \lim_{x\to\pm\infty} \frac{2 + \dfrac{45}{x} + \dfrac{315}{x^2} + \dfrac{600}{x^3}}{1} = 2.$$

OR just use division and

$$\lim_{x\to\pm\infty} \frac{2x^3+45x^2+315x+600}{x^3} = \lim_{x\to\pm\infty} 2 + \frac{45}{x} + \frac{315}{x^2} + \frac{600}{x^3} = 2$$

- First Derivative Information:

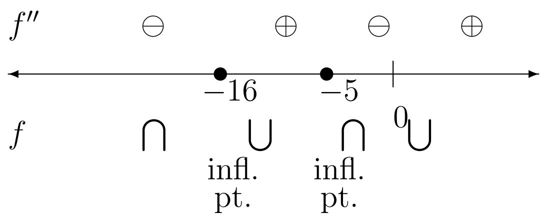
We use the $f'(x)$ given above. The critical numbers are $x = -10$ and $x = -4$. The derivative is undefined at $x = 0$, but that's not a critical number since it was not in the domain of the original function. Using sign testing/analysis for f', around the vertical asymptotes,



So f is increasing on $(-10, -4)$; and f is decreasing on $(-\infty, -10)$, $(-4, 0)$, and $(0, \infty)$. Moreover, f has a local max at $x = -4$ with $f(-4) = \dfrac{17}{16}$ and a local min at $x = -10$ with $f(-10) = \dfrac{1}{20}$.

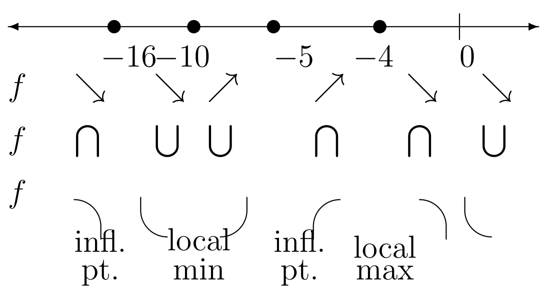
• Second Derivative Information:

Meanwhile, f'' is zero when $x = -16$ and $x = -5$. Using sign testing/analysis for f'' around the vertical asymptote and these possible inflection points $x = -16$ and $x = -5$,

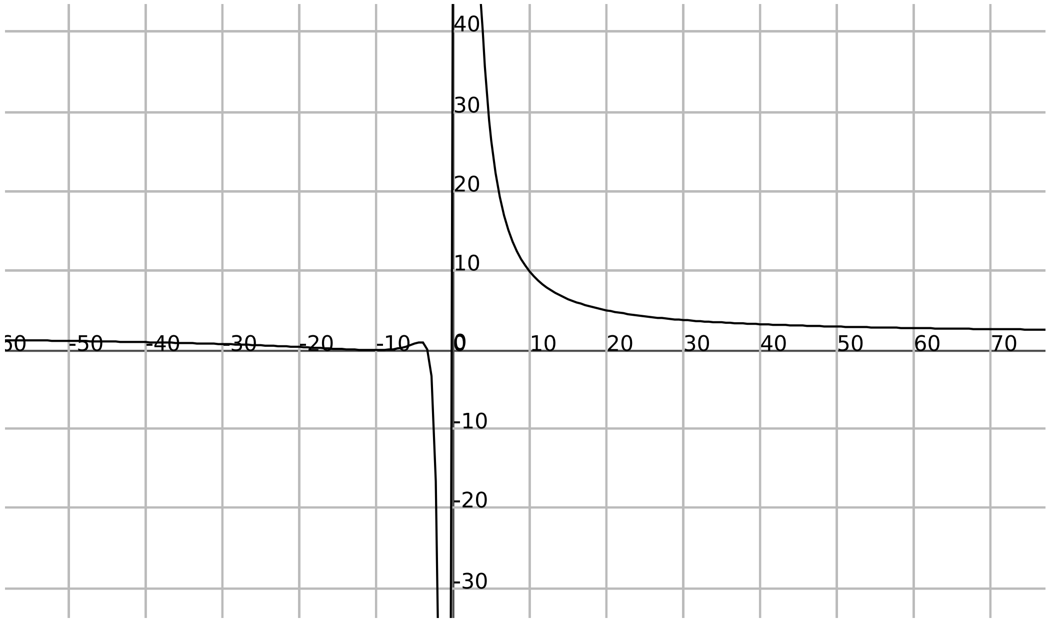


So f is concave down on $(-\infty, -16)$ and $(-5, 0)$, and concave up on $(-16, -5)$ and $(0, \infty)$ with inflection points at $x = -16$ with $f(-16) = \frac{139}{512}$, and at $x = -5$ with $f(-5) = \frac{4}{5}$.

• Piece the first and second derivative information together:



• Sketch:



Position, Velocity, Acceleration

9. A man stands on the edge of a bridge over a river. He throws a stone straight upward in the air with an initial velocity of 64 feet per second. The ball reaches a height of $\mathbf{s(t) = -16t^2 + 64t + 80}$ feet in t seconds above the water. Answer the following questions:

   (a) What is the intitial height of the stone?
   Initial position is $s(0) = 80$ feet.

   (b) What is the maximum height the stone reaches?
   Max height occurs when $v(t) = 0$.
   Compute $v(t) = -32t + 64 \stackrel{\text{set}}{=} 0$ or when $t = 2$ seconds.
   Max height is $s(2) = -64 + 128 + 80 = 144$ feet above the water.

   (c) What is the stone's velocity at time $t = 1$ second?
   The stone's velicity at 1 second is given by $v(1) = -32 + 64 = 32$ ft/sec.

   (d) When is the stone 128 feet above the water?
   Set $s(t) = -16t^2 + 64t + 80 = 128$ and solve for t.
   Here $-16t^2 + 64t - 48 = -16(t^2 - 4t + 3) = -16(t-3)(t-1) = 0$ or when $t = 3$ or $t = 1$ seconds.
   The stone is 128 feet above the water at time 1 second and time 3 seconds.

   (e) What is the stone's acceleration at any time t?
   The stone's acceleration at any time t is given by $a(t) = -32$ ft/sec^2.

   (f) At what time will the stone hit the water?
   The stone hits the water when $s(t) = -16t^2 + 64t + 80 = 0$ which factors

$-16(t^2 - 4t - 5) = -16(t-5)(t+1) = 0$ or when $t = 5$ or $t = -1$ (ignore negative time here).

So the stone hits the water when 5 seconds has passed.

(g) What is the stone's velocity when it hits the water?

If the stone hits the water at 5 seconds, then the velocity at impact is $v(5) = -32(5) + 64 = -160 + 64 = -96$ feet per second.

Turn in your own solutions.