Math 105, Fall 2013

Worksheet 10, Tuesday, November 19, 2013, Answer Key

1. Compute each of the following limits at infinity:

(a) $$\lim_{x\to\infty} \frac{x^2-3x+2013}{x^2+5x-1} = \lim_{x\to\infty} \frac{x^2-3x+2013}{x^2+5x-1} \cdot \frac{\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x^2}\right)}$$

$$= \lim_{x\to\infty} \frac{1-\frac{3}{x}+\frac{2013}{x^2}}{1+\frac{5}{x}-\frac{1}{x^2}} = \frac{1}{1} = \boxed{1}$$

(b) $$\lim_{x\to-\infty} \frac{9x^4-5x^2+7}{5x^4+6x-3} = \lim_{x\to-\infty} \frac{9x^4-5x^2+7}{5x^4+6x-3} \cdot \frac{\left(\frac{1}{x^4}\right)}{\left(\frac{1}{x^4}\right)}$$

$$= \lim_{x\to-\infty} \frac{9-\frac{5}{x^2}+\frac{7}{x^4}}{5+\frac{6}{x^3}-\frac{3}{x^4}} = \boxed{\frac{9}{5}}$$

(c) $$\lim_{x\to\infty} \frac{1-x^3}{7x^3+x^2-100} = \lim_{x\to\infty} \frac{1-x^3}{7x^3+x^2-100} \cdot \frac{\left(\frac{1}{x^3}\right)}{\left(\frac{1}{x^3}\right)}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x^3}-1}{7+\frac{1}{x}-\frac{100}{x^3}} = \boxed{-\frac{1}{7}}$$

(d) $$\lim_{x\to\infty} \frac{x^4+3x^2+6}{x^3+7} = \lim_{x\to\infty} \frac{x^4+3x^2+6}{x^3+7} \cdot \frac{\left(\frac{1}{x^3}\right)}{\left(\frac{1}{x^3}\right)}$$

$$= \lim_{x\to\infty} \frac{x+\frac{3}{x}+\frac{6}{x^3}}{1+\frac{7}{x^3}} = \boxed{+\infty}$$

Note: when the degress of the numerator and denominator do not match, then you can use divide by x to either the power of the denominator or the numerator, but it's usually easier if you use the degree of the denominator.

(e) $\lim_{x\to\infty} \dfrac{x^6 - x^3 + x}{x^7 + x^5 - 9} = \lim_{x\to\infty} \dfrac{x^6 - x^3 + x}{x^7 + x^5 - 9} \cdot \dfrac{\left(\frac{1}{x^7}\right)}{\left(\frac{1}{x^7}\right)}$

$= \lim_{x\to\infty} \dfrac{\frac{1}{x} - \frac{1}{x^4} + \frac{1}{x^7}}{1 + \frac{1}{x^2} - \frac{9}{x^7}} = \dfrac{0}{1} = \boxed{0}$

2. Let $f(x) = x^4 + 4x^3$. For this function, discuss domain, vertical and horizontal asymptote(s), interval(s) of increase or decrease, local extreme value(s), concavity, and inflection point(s). Then use this information to present a detailed and labelled sketch of the curve.

   - Domain: $f(x)$ has domain $(-\infty, \infty)$
   - VA: It is a polynomial, continuous everywhere, and so has no vertical asymptotes.
   - HA: There are no horizontal asymptotes for this f since $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = \infty$ because

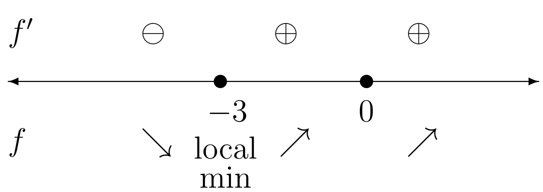
   $\lim_{x\to\infty} x^4 + 4x^3 = \lim_{x\to\infty} x^3(x+4) = \infty \cdot \infty = \infty$ and $\lim_{x\to-\infty} x^4 + 4x^3 = \lim_{x\to-\infty} x^3(x+4) = (-\infty)\cdot(-\infty) = \infty$.

   - First Derivative Information:

   We compute $f'(x) = 4x^3 + 12x^2$ and set it equal to 0 and solve for x to find critical numbers. The critical points occur where f' is undefined (never here) or zero. The latter happens when

$$4x^3 + 12x^2 = 4x^2(x+3) = 0 \Longrightarrow x = 0 \text{ or } x = -3$$

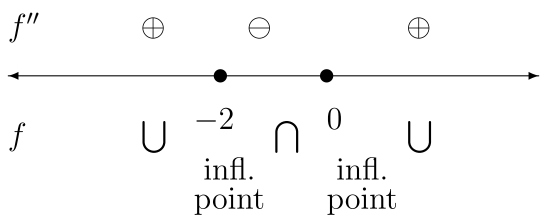
   As a result, $x = 0$ and $x = -3$ are the critical numbers. Using sign testing/analysis for f',



   So f is increasing on $(-3, \infty)$; and f is decreasing on $(-\infty, -3)$. Moreover, f has a local min at $x = -3$ with $f(-3) = -27$.

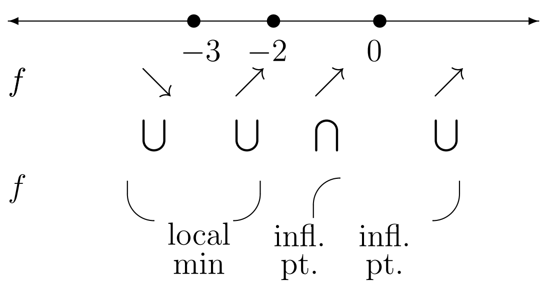
• Second Derivative Information:

Meanwhile, f'' is always defined and continuous, and $f'' = 12x^2 + 24x = 12x(x+2) = 0$ only at our possible inflection points $x = 0$ and $x = -2$. Using sign testing/analysis for f'',

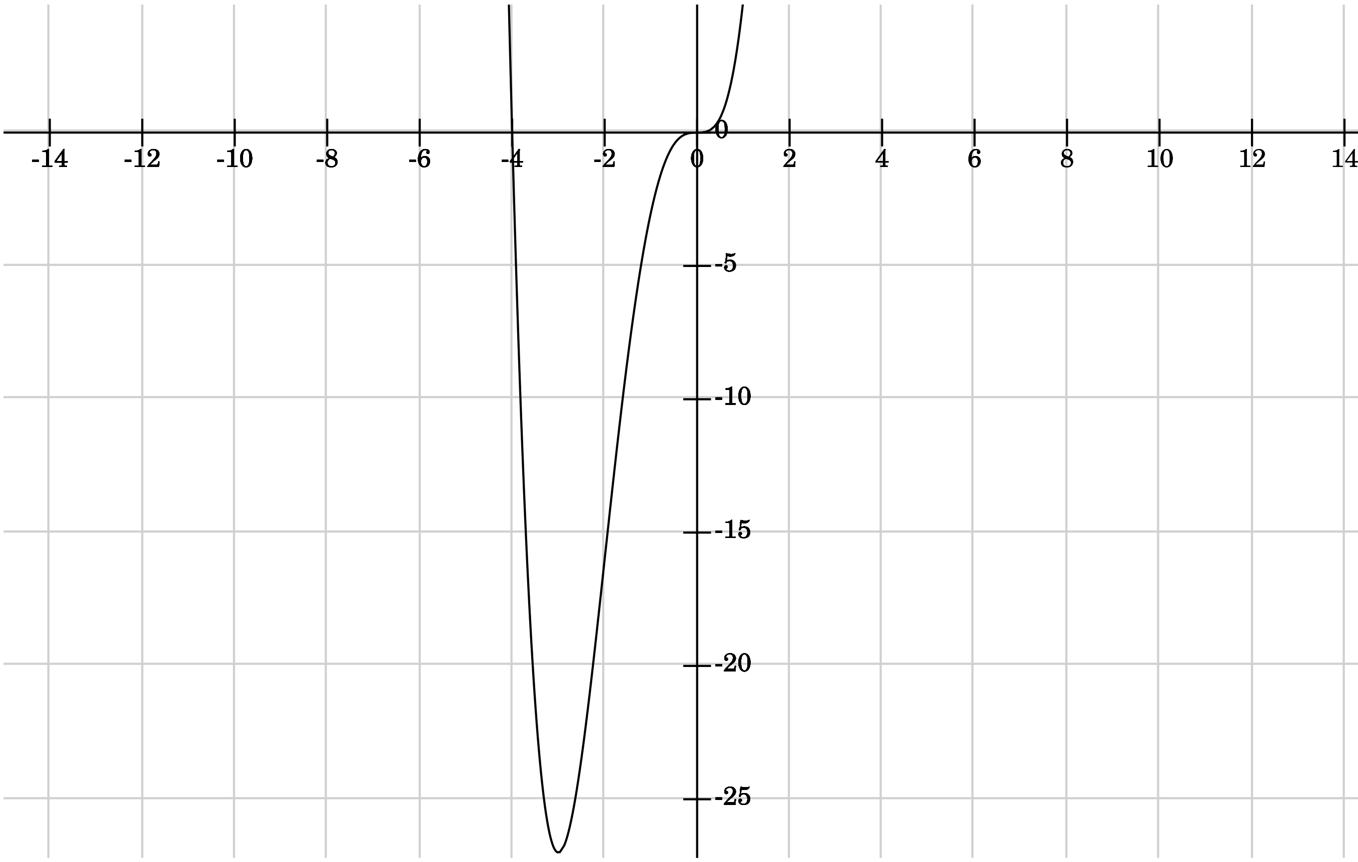


So f is concave down on $(-2, 0)$ and concave up on $(-\infty, -2)$ and $(0, \infty)$, with inflection points at $x = 0$ and $x = -2$ with $f(0) = 0$ and $f(-2) = -16$.

• Piece the first and second derivative information together:



- Sketch:



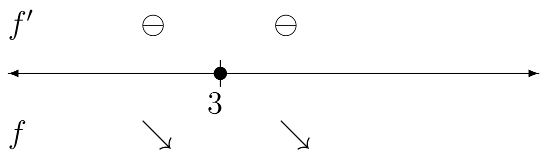
3. Let $f(x) = \dfrac{x}{x-3}$. For this function, discuss domain, vertical and horizontal asymptote(s), interval(s) of increase or decrease, local extreme value(s), concavity, and inflection point(s). Then use this information to present a detailed and labelled sketch of the curve.

   - Domain: $f(x)$ has domain $\{x|x \neq 3\}$
   - VA: Vertical asymptote at $x = 3$.
   - HA: Horizontal asymptote at $y = 1$ for this f since

   $$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{x}{x-3} = \lim_{x\to\pm\infty} \frac{x}{x-3} \cdot \frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} = \lim_{x\to\pm\infty} \frac{1}{1-\frac{3}{x}} = 1$$

   - First Derivative Information:

   We compute $f'(x) = \dfrac{-3}{(x-3)^2}$ to find critical numbers. The critical points occur where f' is zero (never here) or undefined. The latter happens when $x = 3$, which was not in the domain of the original function. As a result, there are techincally no critical numbers. Using sign testing/analysis for f' around the vertical asymptote $x = 3$,

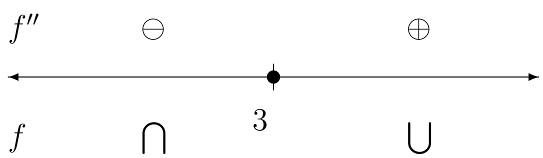


So f is decreasing on $(-\infty, 3)$ and $(3, \infty)$. Moreover, f techincally has no extreme values.

• Second Derivative Information:

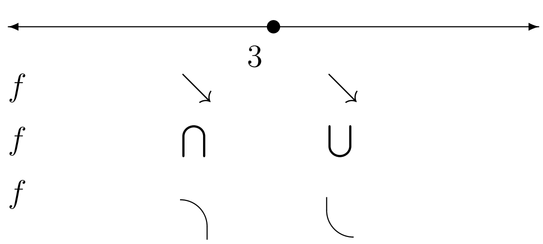
Recall $f'(x) = -3(x-3)^{-2}$.

Meanwhile, $f'' = \dfrac{6}{(x-3)^3}$. No possible inflection points. Using sign testing/analysis for f'' around the vertical asymptote $x = 3$,

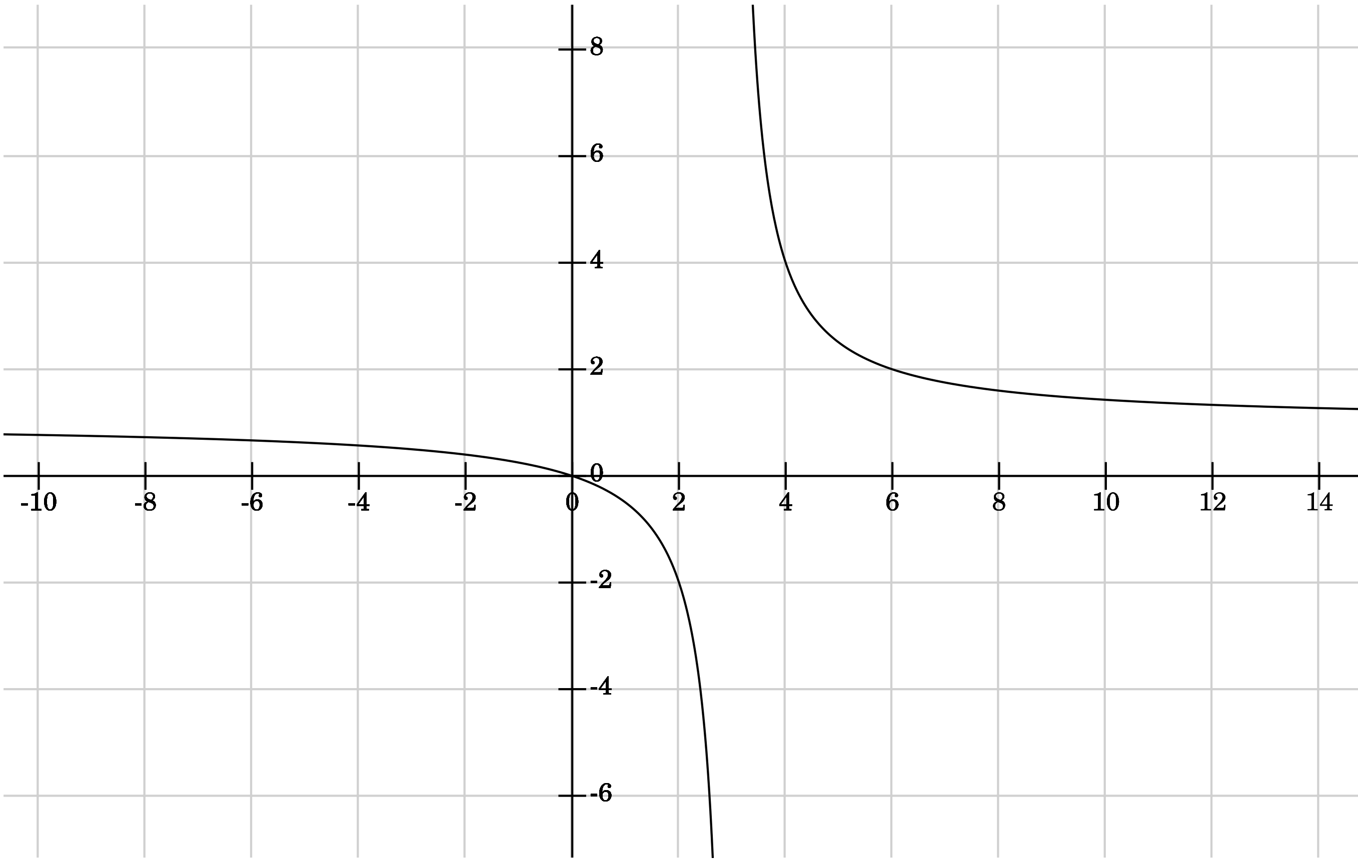


So f is concave down on $(-\infty, 3)$ and concave up on $(3, \infty)$.

• Piece the first and second derivative information together:



• Sketch:

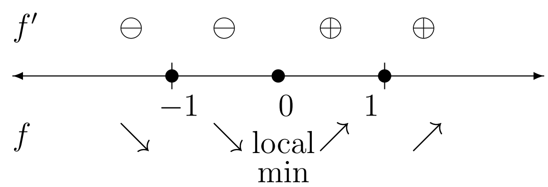


4. Let $f(x) = \dfrac{1}{1-x^2}$. For this function, discuss domain, vertical and horizontal asymptote(s), interval(s) of increase or decrease, local extreme value(s), concavity, and inflection point(s). Then use this information to present a detailed and labelled sketch of the curve.

   - Domain: $f(x)$ has domain $\{x|x \neq \pm 1\}$
   - VA: Vertical asymptotes at $x = \pm 1$.
   - HA: Horizontal asymptote at $y = 0$ for this f since $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \dfrac{1}{1-x^2} = 0$.
   - First Derivative Information:

   We compute $f'(x) = \dfrac{2x}{(1-x^2)^2}$ and set it equal to 0 and solve for x to find critical numbers. The critical points occur where f' is zero or undefined. The former happens when $x = 0$. The derivative is undefined when $x = \pm 1$, but those values are not in the domain of the original function. As a result, $x = 0$ is technically the only critical number.

Using sign testing/analysis for f',



So f is increasing on $(0, 1)$ and $(1, \infty)$; and f is decreasing on $(-\infty - 1)$ and $(-1, 0)$. Moreover, f has a local min at $x = 0$ with $f(0) = 1$.

• Second Derivative Information:

Recall $f'(x) = \dfrac{2x}{(1-x^2)^2}$

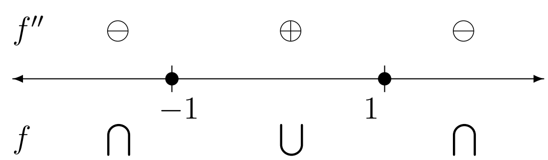
Meanwhile,

$$f'' = \frac{(1-x^2)^2(2) - (2x)2(1-x^2)(-2x)}{(1-x^2)^4} = \frac{2(1-x^2)[(1-x^2)+4x^2]}{(1-x^2)^4}$$

$$= \frac{2(1-x^2)[1+3x^2]}{(1-x^2)^4} = \frac{2[1+3x^2]}{(1-x^2)^3}$$

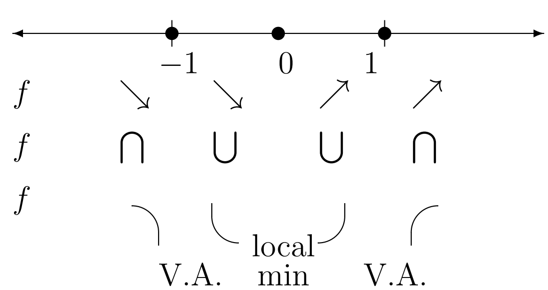
which is never zero since $1 + 3x^2 \neq 0$.

Using sign testing/analysis for f'' around the vertical asymptotes,



So f is concave down on $(-\infty, -1)$ and $(1, \infty)$ and concave up on $(-1, 1)$ with no technical inflection points, since $x = \pm 1$ not in domain of original function.

• Piece the first and second derivative information together:



- Sketch: