Math 105, Fall 2013

Worksheet 9, Tuesday, November 12, 2013   Answer Key

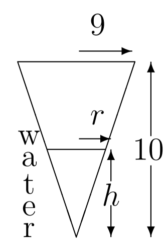
1. A conical tank, 18 feet across the entire top and 10 feet deep, is leaking water at 9 cubic feet per minute. How fast is the height of the water decreasing when the water level is 1 foot?

$$\text{**Recall the volume of the cone is given by } V = \frac{1}{3}\pi r^2 h.$$

NOTE: You should use similar triangles to find a relationship between the radius and the height at general time t.

The cross section (with water level drawn in) looks like:

• Diagram



• Variables
Let r = radius of the water level at time t
Let h = height of the water level at time t
Let V = volume of the water in the tank at time t
Find $\frac{dh}{dt}$ =? when $h = 1$ foot
and $\frac{dV}{dt} = -9\frac{\text{ft}^3}{\text{min}}$

• Equation relating the variables:

Volume= $V = \frac{1}{3}\pi r^2 h$

• Extra solvable information: Note that r is not mentioned in the problem's info. But there is a relationship, via similar triangles, between r and h. We must have

$$\frac{r}{9} = \frac{h}{10} \implies r = \frac{9h}{10}$$

After substituting into our previous equation, we get:

$$V = \frac{1}{3}\pi\left(\frac{9h}{10}\right)^2 h = \frac{81}{300}\pi h^3 = \frac{27}{100}\pi h^3$$

• Differentiate both sides w.r.t. time t.

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{27}{100}\pi h^3\right) \implies \frac{dV}{dt} = \frac{81}{100}\pi h^2 \cdot \frac{dh}{dt} \text{ (Related Rates!)}$$

• Substitute Key Moment Information (now and not before now!!!):

$-9 = \frac{81}{100}\pi(1)^2 \cdot \frac{dh}{dt}$

• Solve for the desired quantity:

$\frac{dh}{dt} = (-9)\left(\frac{100}{81\pi}\right) = -\frac{100}{9\pi}\frac{\text{ft}}{\text{min}}$

• Answer the question that was asked: The height of the water is *decreasing* at a rate of $\frac{100}{9\pi}$ feet every minute at that moment.

2. Write out the definition of a **Critical Number** for a function $f(x)$.

   A number c is a critical number of a function f if it is in the Domain of f and if $f'(c) = 0$ **or** $f'(c)$ Does Not Exist.

3. Find the critical numbers for $f(x) = x\sqrt{1-x}$.

   First compute the derivative

   $f'(x) = x\frac{1}{2\sqrt{1-x}}(-1) + \sqrt{1-x}(1) = \frac{-x+2-2x}{2\sqrt{1-x}} = \frac{-3x+2}{2\sqrt{1-x}}$.

   Critical numbers are where the derivative equals 0 or is undefined.

   First $f'(x) = \frac{-3x+2}{2\sqrt{1-x}} = 0$ when the numerator equals 0, which is when $x = \frac{2}{3}$

   Second the derivative is undefined here where the denominator is 0, which is when $x = 1$, which we should note **is** in the domain of the original function.

   Finally the critical numbers are $\boxed{x = \frac{2}{3}}$ and $\boxed{x = 1}$.

4. Find the critical numbers for $f(x) = \frac{x^2+1}{x+3}$

   First $f'(x) = \frac{(x+3)(2x) - (x^2+1)(1)}{(x+3)^2} = \frac{2x^2+6x-x^2-1}{(x+3)^2} = \frac{x^2+6x-1}{(x+3)^2} = 0$

   when the numerator equals 0, which is when $x^2 + 6x - 1 = 0$.

   Using the quadratic formula, we solve $x = \frac{-6 \pm \sqrt{36+4}}{2} = \frac{-6 \pm \sqrt{40}}{2} = \frac{-6 \pm 2\sqrt{10}}{2} = -3 \pm \sqrt{10}$

   Secondly, the derivative is undefined when $x = -3$, but note that $x = -3$ was never in the domain of the original function, so it can not technically be a critical number.

   Finally the critical numbers are $\boxed{x = -3 + \sqrt{10}}$ and $\boxed{x = -3 - \sqrt{10}}$.

5. Find the critical numbers for $f(x) = x^{\frac{3}{4}} - 2x^{\frac{1}{4}}$

   First $f'(x) = \frac{3}{4}x^{-\frac{1}{4}} - 2\left(\frac{1}{4}\right)x^{-\frac{3}{4}} = \frac{3}{4x^{\frac{1}{4}}} - \frac{1}{2x^{\frac{3}{4}}} = \frac{3x^{\frac{1}{2}}}{4x^{\frac{3}{4}}} - \frac{2}{4x^{\frac{3}{4}}} = \frac{3x^{\frac{1}{2}} - 2}{4x^{\frac{3}{4}}} = 0$

   when the numerator equals 0, which is when $3\sqrt{x} - 2 = 0$ or when $\sqrt{x} = \frac{2}{3}$ or when $x = \frac{4}{9}$.

   Secondly the derivative is underfined when the denominator equals 0 here, when $x = 0$.

   Finally the critical numbers are $\boxed{x = \frac{4}{9}}$ and $\boxed{x = 0}$

Recall: the **Extreme Value Theorem** states that a continuous function on a bounded interval is guaranteed **both** an absolute maximum and minimum value on that interval.

Question: How do we find those values?

Answer: We use the Closed Interval Method.

Closed Interval Method Supposed that a function f is a continuous function on a closed/bounded interval. To find the absolute maximum and minimum value(s) of f:

Step 1: Find the crticial numbers for f in the interval.

Step 2: Evaluate the function f at the critical numbers from Step 1.

Step 3: Evaluate the function f at the endpoints.

Step 4: Pick off the largest of these values from Steps 2 and 3, as the Absolute Maximum Value. Then pick off the smallest of these values in Steps 2 and 3, as the Absolute Minimum Value.

6. Find the absolute maximum and absolute minumum value(s) of the function

   $$F(x) = x^3 - 3x^2 \text{ on the interval } [-1, 1].$$

   $F'(x) = 3x^2 - 6x = 3x(x - 2)$ Simplify fully.

   $F'(x) \overset{\text{set}}{=} 0$ when $x = 0$ or $x = 2$.

   $F'(x)$ is always defined here, since it's a polynomial.

   So the critical numbers are $x = 0$ and $x = 2$, but $x = 2$ is NOT in the interval of interest here.

   Applying the Closed Interval method:

   $F(0) = \boxed{0} \longleftarrow$ Absolute Maximum Value

   $F(1) = -2$

   $F(-1) = \boxed{-4} \longleftarrow$ Absolute Minimum Value

   So the absolute maximum value is 0 (attained at $x = 0$), and the absolute minimum value is -4 (attained at $x = -1$).

7. Find the absolute maximum and absolute minumum value(s) of the function

$$G(x) = (x-1)^2(x-9)^2 \text{ on the interval } [0, 8].$$

$G'(x) = (x-1)^2 \cdot 2(x-9) + (x-9)^2 \cdot 2(x-1) = 2(x-1)(x-9)(2x-10)$. On the interval $[0, 8]$, G' is always defined. Also, $G'(x) = 0$ happens only when $x = 1$, $x = 9$, and $x = 5$ (our critical numbers). Here $x = 9$ is outside of our interval of interest. Applying the closed interval method:

$G(1) = \boxed{0} \longleftarrow$ Absolute Minimum Value

$G(5) = \boxed{256} \longleftarrow$ Absolute Maximum Value

$G(0) = 81$

$G(8) = 49.$

So the absolute maximum value is 256 (attained at $x = 5$), and the absolute minimum value is 0 (attained at $x = 1$).

8. Find the absolute maximum and absolute minumum value(s) of the function

$$H(x) = \frac{10x}{x^2+1} \text{ on the interval } [0, 2].$$

First $H'(x) = \dfrac{(x^2+1)10 - 10x(2x)}{(x^2+1)^2} = \dfrac{10x^2+10-20x^2}{(x^2+1)^2} = \dfrac{-10x^2+10}{(x^2+1)^2} = \dfrac{10-10x^2}{(x^2+1)^2}$

So critical numbers are when $10 - 10x^2 = 0$ or when $x = \pm 1$. Notice that $x = -1$ is not in our interval of interest, so we evaluate H at $x = 1$ here and at the endpoints $x = 0$ and $x = 2$ and use the Closed Interval Method:

$H(1) = \dfrac{10}{2} = \boxed{5} \longleftarrow$ Absolute Maximum Value

$H(0) = \boxed{0} \longleftarrow$ Absolute Minimum Value

$H(2) = \dfrac{20}{5} = 4$