Math 105, Fall 2013

Worksheet 8, Tuesday, November 5, 2013, Answer Key

Reminder: This worksheet is a chance for you not to just *do* the problems, but rather *understand* the problems. Please discuss ideas with your partners. Your solutions should be focused moreso on presentation than on numerical values. For your Related Rates problems, you should double-check if your answer *makes sense*. If your quantity is decreasing or shrinking, then you should get a negative rate of change. (Why?) If your quantity is increasing or growing, then you should get a positive rate of change. (Why?)

Position-Velocity Problems

1. A stone is dropped from a bridge that is 576 feet above a river. The stone's position above the water is given in feet at time t by $s(t) = -16t^2 + 576$.

   - How long does it take for the stone to impact the water (fixed at position 0 here)?

   The stone impacts the water when $s(t) = 0 = -16t^2 + 576$. That is, when $t^2 = \dfrac{576}{16} = 36$. Thus, $t = \pm 6$, and we will only consider the positive time $t = 6$ seconds here. The stone impacts the water at $\boxed{t = 6}$ seconds.

   - What is the stone's velocity when it impacts the water?

   The velocity is given by $v(t) = s'(t) = -32t$. So the velocity of the stone when it impacts the water, at $t = 6$ seconds, is given by $v(6) = -32(6) = \boxed{-192}\,\frac{\text{ft}}{\text{sec}}$.

2. Suppose a falling balls position is given by $s(t) = 256 - 16t^2$ feet at t seconds.

   - What is the balls initial position above the ground?

   Initial position is $\boxed{s(0) = 256}$ feet.

   - Find the average velocity of the ball during the initial two s econds of its drop.

   Average velocity= $V_{\text{ave}} = \dfrac{\text{change in distance}}{\text{change in time}} = \dfrac{s(2) - s(0)}{2 - 0} = \dfrac{(256 - 16(2)^2) - 256}{2} = \dfrac{-64}{2} = \boxed{-32}\,\dfrac{\text{ft}}{\text{sec}}$

   - Find the velocity at 2 seconds and 3 seconds respectively.

   $v(t) = s'(t) = -32t$ so $\boxed{v(2) = -64}\,\frac{\text{ft}}{\text{sec}}$ and $\boxed{v(3) = -96}\,\frac{\text{ft}}{\text{sec}}$

   Understand why the velocity is negative here. The ball is falling in the negatively oriented direction. So the position $s(t)$ is decreasing $\searrow$ which means its derivative $s'(t) = v(t)$ is negative.

   - How much time passed before the ball hit the ground?

   First find when the ball hits the ground. Set $s(t) = 0$ and solve for time t.

   $s(t) = 256 - 16t^2 = 0 \implies t^2 = \frac{256}{16} \implies t = \pm 4 \implies \boxed{t = 4}$ seconds, since we are considering positive time here. So the ball hits the ground in 4 seconds.

• What was the balls velocity when it hit the ground?

The ball's velocity, when it hits the ground, is $v(4) = -32(4) = \boxed{-128}\frac{\text{ft}}{\text{sec}}$

• Finally, find the balls acceleration at 3 secon

$a(t) = v'(t) = s''(t) = -32$ (note, it's constant here, acceleration due to gravity)

So, $\boxed{a(3) = -32}\frac{\text{ft}}{\text{sec}^2}$

Related Rates: For the following problems I expect you to follow the guidelines set in class. You must show your work outlined as follows, with full labelling:

• Diagram

• Variables

• Given Information

• Equation relating variables

• Differentiate

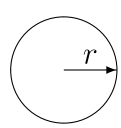
• Substitute (given information)

• Solve (for desired quantity or rate)

• Answer the original questions (in words!!)

3. Suppose a snowball remains spherical while it melts with the radius shrinking at one inch per hour. How fast is the volume of the snowball decreasing when the radius is 2 inches?

• Diagram(cross-sectioned in 2 dimensions here)



• Variables

Let r = radius of the sphere at time t

Let V = volume of the sphere at time t

Find $\frac{dV}{dt}$ =? when $r = 2$ feet

and $\frac{dr}{dt} = -1\frac{\text{in}}{\text{hr}}$

• Equation relating the variables:

Volume $V = \frac{4}{3}\pi r^3$

• Differentiate both sides w.r.t. time t.

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) \implies \frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \cdot \frac{dr}{dt} \implies \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \text{ (Related Rates!)}$$

• Substitute Key Moment Information (now and not before now!!!):

$\frac{dV}{dt} = 4\pi(2)^2(-1)$

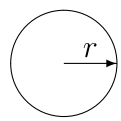
• Solve for the desired quantity:

$\frac{dV}{dt} = -16\pi \frac{\text{in}}{\text{hr}}$

• Answer the question that was asked: The volume of the snowball is *decreasing* at a rate of 16π inches every hour at that moment.

4. An oil spill occurs at sea. The oil gushes out from an offshore derrick and forms a circle whose area increases at a rate of 100 ft^2/min. How fast is the radius of the spill increasing when the spill is 20 feet across?

• Diagram



• Variables

Let r = radius of the spill at time t

Let A = area of the spill at time t

Find $\frac{dr}{dt}$ =? when diameter $= 20 \Rightarrow r = 10$ ft

and $\frac{dA}{dt} = 100 \frac{\text{ft}^2}{\text{min}}$

• Equation relating the variables:

Area $A = \pi r^2$.

• Differentiate both sides w.r.t. time t.

$\frac{d}{dt}(A) = \frac{d}{dt}(\pi r^2) \implies \frac{dA}{dt} = 2\pi r \frac{dr}{dt}$ (Related Rates!)

• Substitute Key Moment Information (now and not before now!!!):

$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

$100 = 2\pi(10)\frac{dr}{dt} = 20\pi \frac{dr}{dt}$

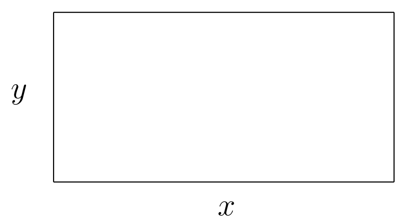
• Solve for the desired quantity:

$\frac{dr}{dt} = \frac{100}{20\pi} = \frac{5}{\pi} \frac{\text{ft}}{\text{min}}$

• Answer the question that was asked: The radius of the oil spill is *increasing* at a rate of $\frac{5}{\pi}$ feet every minute at that moment.

5. The sides of a rectangle change with respect to time. The width is increasing at a rate of 2 in/sec. while the length is decreasing at a rate of 3 in/sec. How fast is the area of the rectangle changing when the width is 6 inches and the length is 8 inches?

• Diagram



• Variables

Let $x =$ length of the rectangle at time t

Let $y =$ width of the rectangle at time t

Let $A =$ the area of the rectangle at time t

Find $\frac{dA}{dt} = ?$ when $x = 8$ inches, $y = 6$ inches, $\frac{dx}{dt} = -3$ in/sec and $\frac{dy}{dt} = 2$ in/sec

• Equation relating the variables:

The Area of the rectangle is given by length times width

$A = xy$.

• Differentiate both sides w.r.t. time t.

$\frac{d}{dt}(A) = \frac{d}{dt}(xy) \implies \frac{dA}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt}$ (Related Rates!)

• Substitute Key Moment Information (now and not before now!!!):

$\frac{dA}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt} = 8(2) + 6(-3)$

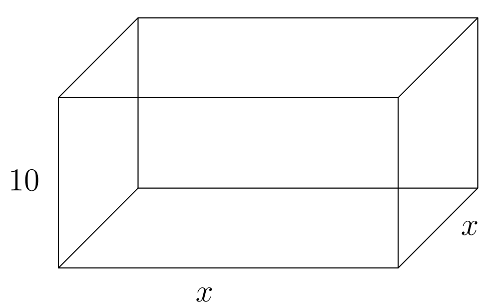
• Solve for the desired quantity:

$\frac{dA}{dt} = 16 - 18 = -2 \text{ in}^2/\text{sec}$.

• Answer the question that was asked: The area is *decreasing* at a rate of 2 inches per second at that moment.

6. A box has a square base, and its height is always 10 inches. If the edge of the base is increasing at a rate of 2 in/min, how fast is the volume of the box increasing when the edge is 8 inches?

• Diagram



- Variables

Let x = length of base of box at time t
Let V = volume of the box at time t
Find $\frac{dV}{dt}$ =? when $x = 8$ in, and $\frac{dx}{dt} = 2$ in/min

- Equation relating the variables:

We use the volume formula for a box. Volume equals length times width times height. Here we have the length equal to the width, since it's a square base.

$V = 10x^2$.

- Differentiate both sides w.r.t. time t.

$\frac{d}{dt}(V) = \frac{d}{dt}(10x^2) \implies \frac{dV}{dt} = 20x\frac{dx}{dt}$ (Related Rates!)

- Substitute Key Moment Information (now and not before now!!!):

$\frac{dV}{dt} = 20x\frac{dx}{dt} = 20(8)(2)$

- Solve for the desired quantity:

$\frac{dV}{dt} = 320 \text{ in}^3/\text{min}$

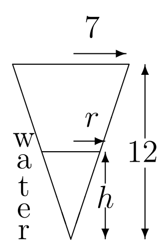
- Answer the question that was asked: The volume of the box is *increasing* at a rate of 320 cubic inches every minute at that moment.

7. A conical tank, 14 feet acorss the entire top and 12 feet deep, is leaking water. The radius of the water level is decreasing at the rate of 2 feet per minute. How fast is the water leaking out of the tank when the radius of the water level is 2 feet?

*** Recall the volume of the cone is given by $V = \frac{1}{3}\pi r^2 h$.

The cross section (with water level drawn in) looks like:

- Diagram



- Variables

Let r = radius of the water level at time t
Let h = height of the water level at time t
Let V = volume of the water in the tank at time t
Find $\frac{dV}{dt}$ =? when $r = 2$ feet
and $\frac{dr}{dt} = -2\frac{\text{ft}}{\text{min}}$

• Equation relating the variables:

Volume= $V = \frac{1}{3}\pi r^2 h$

• Extra solvable information: Note that h is not mentioned in the problem's info. But there is a relationship, via similar triangles, between r and h. We must have

$$\frac{r}{7} = \frac{h}{12} \implies h = \frac{12r}{7}$$

After substituting into our previous equation, we get:

$V = \frac{1}{3}\pi r^2 \left(\frac{12r}{7}\right) = \frac{4}{7}\pi r^3$

• Differentiate both sides w.r.t. time t.

$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{4}{7}\pi r^3\right) \implies \frac{dV}{dt} = \frac{12}{7}\pi r^2 \cdot \frac{dr}{dt}$ (Related Rates!)

• Substitute Key Moment Information (now and not before now!!!):

$\frac{dV}{dt} = \frac{12}{7}\pi r^2 \cdot \frac{dr}{dt} = \frac{12}{7}\pi (2)^2(-2)$

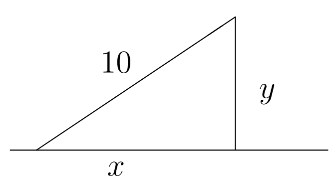
• Solve for the desired quantity:

$\frac{dV}{dt} = -\frac{96\pi}{7} \frac{\text{ft}^3}{\text{min}}$

• Answer the question that was asked: The water is *leaking out* at a rate of $\frac{96\pi}{7}$ cubic feet every minute at that moment.

8. The top of a ten foot ladder is sliding down a vertical wall at the rate of one foot every second. How fast is distance between the bottom of the ladder and the wall changing when the top of the ladder is three feet above the ground?

   • Diagram



• Variables

Let x = distance between bottom of ladder and wall at time t

Let y = distance between top of ladder and ground at the deep end at time t

Find $\frac{dx}{dt}$ =? when $y = 3$ ft

and $\frac{dy}{dt} = -1\frac{\text{ft}}{\text{sec}}$

• Equation relating the variables:

We have $x^2 + y^2 = (10)^2$.

Or $x^2 + y^2 = 100$

• Differentiate both sides w.r.t. time t.

$$\frac{d}{dt}(x^2 + y^2) = \frac{d}{dt}(100) \implies 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \text{ (Related Rates!)}$$

• Extra solvable information: Here we are not given the length of the base at the key moment, but we can use the Pythagorean Theorem to compute that value.

$x^2 + y^2 = 100$

so $x = \sqrt{100 - y^2} = \sqrt{100 - 9} = \sqrt{91}$.

• Substitute Key Moment Information (now and not before now!!!):

$$2\sqrt{91}\frac{dx}{dt} + 2(3)(-1) = 0$$

$$2\sqrt{91}\frac{dx}{dt} - 6 = 0$$

• Solve for the desired quantity:

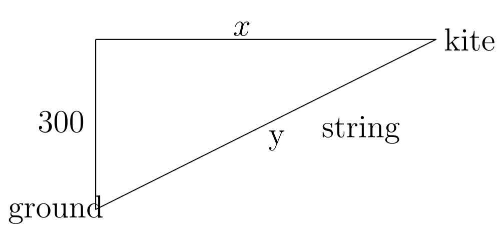
$$\frac{dx}{dt} = \frac{6}{2\sqrt{91}} = \frac{3}{\sqrt{91}}$$

• Answer the question that was asked: The distance between the bottom of the ladder and the wall is *increasing* at a rate of $\frac{3}{\sqrt{91}}$ feet every second at that moment.

9. A kite starts flying 300 feet directly above the ground. The kite is being blown horizontally at 10 feet per second. When the kite has blown horizontally for 40 seconds, how fast is the string running out?

(a) • Diagram

The picture at arbitrary time t is:



• Variables

Let x = distance kite has travelled horizontally at time t

Let y = length of the string let out at time t

Find $\frac{dy}{dt}$ =? when $t = 40$ sec.

and $\frac{dx}{dt} = 10\frac{\text{ft}}{\text{sec}}$

• Equation relating the variables:

We have $x^2 + (300)^2 = y^2$ by the Pythagorean Theorem.

• Differentiate both sides w.r.t. time t.

$$\frac{d}{dt}(x^2 + (300)^2) = \frac{d}{dt}(y^2) \implies 2x\frac{dx}{dt} = 2y\frac{dy}{dt} \implies x\frac{dx}{dt} = y\frac{dy}{dt} \text{ (Related Rates!)}$$

• Extra solvable information:

At the key instant when $t = 40$, using the original rate of the horizontal travel of the kite, we have

Distance =Rate times Time.

So at that moment, 40 seconds later $x = \frac{dx}{dt}(t) = (10)(40) = 400$ feet.

We also need to find the length of the string at that key moment:

We can use the Pythagorean Theorem again

$y = \sqrt{(300)^2 + (400^2} = \sqrt{90000 + 160000} = \sqrt{250000} = 500$

or you could use the 3-4-5 triangle rule.

• Substitute Key Moment Information (now and not before now!!!):

$$(400)(10) = (500)\frac{dy}{dt}$$

• Solve for the desired quantity:

$$\frac{dy}{dt} = \frac{4000}{500} = 8\frac{\text{ft}}{\text{sec}}$$

• Answer the question that was asked: The string is *running out* at a rate of 8 feet per second at that moment.